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Preserving order observers for nonlinear systems

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SUMMARY

Preserving Order Observers provide an estimation that is always above or below the true variable and, in the absence of uncertainties/perturbations, the estimation converges asymptotically to the true value of the variable. In this paper we propose a novel methodology to design *preserving order observers* for a class of nonlinear systems in the nominal case or when perturbations/uncertainties are present. This objective is achieved by combining two important systemic properties: *dissipativity* and *cooperativity*. Dissipativity is used to guarantee the convergence of the estimation error dynamics, while cooperativity of the error dynamics assures the order preserving properties of the observer. The use of dissipativity for observer design offers a big flexibility in the class of nonlinearities that can be considered while keeping the design simple: it leads in many situations to the solution of a Linear Matrix Inequality (LMI). Cooperativity of the observer leads to a LMI. When both properties are considered simultaneously the design of the observer can be reduced, in most cases, to the solution of both a Bilinear Matrix Inequality (BMI) and a Linear Matrix Inequality (LMI). Since a couple of preserving order observers, one above and one below, provide an *interval observer*, the proposed methodology unifies several interval observers design methods. The design methodology has been validated experimentally in a three-tanks system, and it has also been tested numerically and compared to an example from the literature. Copyright © 2012 John Wiley & Sons, Ltd.

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KEY WORDS: Interval Observers; Dissipative Observers; Cooperative Systems

1. INTRODUCTION

Observers provide a state variables estimation which converges asymptotically to their true values, at least when the plant model is perfectly known and/or no unknown disturbances are acting on the system. One drawback for certain applications is the fact that during the time span of convergence it is not possible to trust in the estimation given by the observer. Decisions taken on the basis of such estimation can lead to bad behavior in control (recall the use of saturating functions to avoid the destabilizing effect in output feedback), or to wrong decisions in failure detection.

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This situation is even worse when (non vanishing) uncertainties and/or perturbations are present. Since in this case it is not possible to construct a continuous observer without error, an alternative in many applications would be to have an estimation of a state variables that is always above or below the true value. This would allow, for example, to send an alarm signal when the temperature of a nuclear reactor is about to reach the maximal critical value before it really reaches it, if the estimated temperature would be guaranteed above the true temperature. Such behavior can be obtained by a *Preserving Order Observer*, defined as an observer whose estimates always stay above or below the true state trajectory and, in the absence of uncertainites/perturbations, the estimation error converges asymptotically to zero. Preserving order (or monotone) systems have very important properties, and they have been studied since several decades in mathematics and control [2, 14], of which Cooperative systems build an important subclass. Although monotone systems have found a growing interest in modeling and control, relatively few works on its use for observation purposes have appeared.

Interval observers, which consist of a couple of preserving order observers, one giving an upper and the other a lower estimation of the state, have been proposed in the literature to provide guaranteed bounds at any instant of time for the states of an uncertain dynamical system. The first application of the preserving order observers appeared in [12, 25], where interval observers are introduced and applied to a class of nonlinear systems with uncertainties. They have been mainly applied to the estimation of parameters or non measurable variables in biological systems, for which observation issues are very challenging due to the limited availability of on-line sensors and the uncertainties related to the model dynamics. Experimental validation of Interval Observers has been also reported in [1] for highly uncertain bioreactors. A combination of interval observers and the so called asymptotic observers [5, 9, 6] is proposed in [26], where the state observation for bioprocesses under uncertain process parameters and/or process inputs is solved, without requiring the knowledge of the process kinetics.

One interesting feature of interval observers is that they allow the comparison of the estimates of several interval observers: the best upper bound is the lowest of the upper bounds, and vice versa for the lower bounds. This has lead to the development of bundles of interval observers [6, 19], which consist of several interval observers running in parallel and the best estimate is taken at each time instant. In these bundles some estimates can be unstable. In this case the preserving order observers are called *framers*, since they do not have any stability properties. Recently in [20] a robust interval observer to estimate the unknown variables of uncertain chaotic systems is presented.

Since cooperativity is a coordinate dependent property, it is possible to use coordinate transformations to obtain interval observers in appropriate coordinates. This idea has been used recently in [17, 18] to show that it is possible to design an exponentially convergent interval observer for any Linear Time Invariant System, with additive perturbations, by using a linear *time-varying* coordinate transformation.

The objective of this paper is to pursue the line of research initiated in [12] (see also [19, 20]), for designing interval observers in the original coordinates of the system. The main idea is to merge the dissipative observer design methodology, introduced by one of the authors several years ago [21, 22, 23, 27], with the basic idea of making the observer error system a cooperative one (see [4]), so that the observer preserves the order and converges to the true values, in the nominal case, i.e. in the absence of perturbations or uncertainties. For the perturbed case, the convergence will be weakened to the practical one: the observation error is Input-to-State Stable with respect to the perturbing signal. Moreover, the estimation error is required to be cooperative with respect to the perturbation. One of the advantages of using the dissipative observer design method resides in the fact that it is very flexible and unifies several observer design methods known to date, as for example the High-Gain Observer or the Circle-Criterion Observer design methods [21, 22, 23, 27].

Thus we consider a class of nonlinear systems, in absence and in presence of perturbations, for which dissipative observers can be designed, and a preserving order property will be imposed to render the estimation error of the observer cooperative. These systems are an extension of those proposed in [12], where interval observers were introduced. A first step consists in taking the error dynamics and to decompose it into a linear time invariant subsystem and a nonlinear time varying feedback. If the nonlinearity is dissipative with respect to a quadratic supply rate, the linear part must be designed to be dissipative with respect to a related supply rate, to assure exponential stability of the closed loop [21, 22, 23, 27]. Likewise, if the observation error dynamics is a cooperative system (with respect to the uncertainties/perturbations in case they exist) then the trajectories of the error are assured to preserve the order and therefore the observer estimate dynamically upper or lower bounds for the states, depending on the order of the initial error. Therefore, in a second step the estimation error will be made cooperative. Thus, *preserving order observers for nonlinear systems* fulfill the two characteristics: observers preserve the order and the error estimation converges to zero in the absence of uncertainties/perturbations, or the error will be ultimately bounded for the perturbed case. These estimators were defined as cooperative observers in [4]. To build an interval observer two preserving order observers will be run in parallel: one provides an upper while the other a lower estimation of the states. The design of these observers can be reduced, in most cases, to a linear matrix inequality (LMI) and a bilinear matrix inequality (BMI), which are standard tools in control theory. The performance of the preserving order observers has been tested in several numerical simulations.

2. PRELIMINARIES

In this work two system properties will be used for the design of preserving order observers: i) cooperativity is an order preserving property [28, 2, 14], and it will be fundamental for the interval observers, and ii) dissipativity [8, 31, 32] (see also [15] and [21]) will be used to assure the convergence properties of the estimation error. Some relevant results in these fields will be recalled here.

2.1. *Cooperative Systems*

The symbol $\succeq$ represents a *partial order* in a space of vectors or matrices. For vectors $x, y \in \mathbb{R}^n$, $\; x \succeq y \Leftrightarrow x_i - y_i \geq 0, \forall\; i \in \{1, \ldots, n\}$, i.e. each component of x is greater or equal to the corresponding component of y. For matrices $M, N \in \mathbb{R}^{n\times m}$, $\; M \succeq N \Leftrightarrow M_{ij} - N_{ij} \geq 0, \;, \;\forall\; i, j \in \{1, \ldots, n\}$. In particular, *non negative* vectors or matrices satisfy, $x \succeq 0 \Leftrightarrow x_i \geq 0, \;\forall\, i \in \{1, \ldots, n\}$ or $M \succeq 0 \Leftrightarrow M_{ij} \geq 0, \;\forall\, i, j \in \{1, \ldots, n\}$, respectively.

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Cooperative systems [2, 14], which are a special class of monotone systems, are *those whose state and output trajectories preserve the partial order at every time, when the input signals (if they are present) and the initial states are (partially) ordered.*

Definition 1
Consider a nonlinear system

$$\Sigma_{NL} \left\{ \begin{array}{l} \dot{x} = F\left(t, x, u\right), \quad x(0) = x_0 \\ y = H\left(t, x, u\right) \ , \end{array} \right. \tag{1}$$

where $x \in \mathbb{R}^n$ is the *state*, $u \in \mathbb{R}^m$ is the *input*, and $y \in \mathbb{R}^p$ is the *output* of the system. F is a smooth vector field and H is a smooth nonlinear function, both depending on the variables (t, x, u). We will assume that the system is *complete*, i.e. the trajectories exist for all positive times $t \geq t_0$. Σ_{NL} is *cooperative* if whenever the initial states and the inputs are ordered, i.e.

$$x_0^1 \succeq x_0^2, \;\; u^1\left(t\right) \succeq u^2\left(t\right), \;\; \forall t \geq 0$$

it follows that the state and the output trajectories are ordered too, i.e.

$$x\left(t, t_0, x_0^1, u^1\left(t\right)\right) \succeq x\left(t, t_0, x_0^2, u^2\left(t\right)\right) \ ,$$

$$H \circ x\left(t, t_0, x_0^1, u^1\left(t\right)\right) \succeq H \circ x\left(t, t_0, x_0^2, u^2\left(t\right)\right) \ , \forall t \geq t_0 \, .$$

Note that when no inputs and outputs are present, the previous definition reduces to the classical one, in which ordered initial states implies ordered state trajectories [28]. Smooth cooperative systems can be characterized in a simple way.

Proposition 2 ([2])
The system Σ_{NL} in (1) is *cooperative* if and only if all the following conditions are satisfied:

$$\textbf{(C1)} \quad \left[\frac{\partial F_i}{\partial x_j}\right] \overset{M}{\succeq} 0 \Leftrightarrow \frac{\partial F_i}{\partial x_j} \geq 0, \, \forall i \neq j \, ; \qquad \textbf{(C2)} \quad \left[\frac{\partial F_i}{\partial u_j}\right] \succeq 0 \, ; \qquad \textbf{(C3)} \quad \left[\frac{\partial H_i}{\partial x_j}\right] \succeq 0 \, . \qquad \square$$

The symbol $\overset{M}{\succeq} 0$ represents that $\left[\frac{\partial F_i}{\partial x_j}\right]$ is *Metzler*, that is, the off-diagonal elements are non negative. In the classical situation without inputs/outputs cooperativity is equivalent to condition (C1). Linear cooperative systems are specially easy to characterize:

Proposition 3 ([2])
Consider the linear continuous time system

$$\Sigma_L : \left\{ \begin{array}{l} \dot{x} = Ax + Bu \, , \quad x\left(0\right) = x_0 \\ y = Cx \, , \end{array} \right. \tag{2}$$

where $(x, u, y) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$ are the state, the input and the output vectors, respectively. The system Σ_L in (2) is *cooperative* if and only if,

$$\textbf{(C1)} \quad A \overset{M}{\succeq} 0 \Leftrightarrow a_{ij} \geq 0, \; \forall i \neq j \, ; \qquad \textbf{(C2)} \quad B \succeq 0 \, ; \qquad \textbf{(C3)} \quad C \succeq 0 \, . \qquad \square$$

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We recall that *Positivity* is a related property to cooperativity. A system is positive if whenever the input and the initial conditions are non negative the state and output trajectories will be also non negative, i.e. $x_0 \succeq 0$ and $u(t) \succeq 0$ implies that $x(t, t_0, x_0, u(t)) \succeq 0$ and $H \circ x(t, t_0, x_0, u(t)) \succeq 0$. For Linear Systems cooperativity and positivity are equivalent properties, but this is not the case for nonlinear systems.

2.2. *Dissipative Systems*

As dissipativity will play a fundamental role in the convergence of the observers, some results (see [31, 32, 8, 21, 22, 23, 27]) are recalled.

Consider a (smooth) nonlinear system Σ_{NL} in (1), where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^p$ the state, the input and the output vectors, respectively. Assume that $F(t,0,0) = 0$ and $H(t,0,0) = 0$. A function $w(y,u) : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$, such that $w(0,0) = 0$ is called a *supply rate* for Σ_{NL} if $w(y,u)$ is locally integrable for all input-output pairs of Σ_{NL}. Σ_{NL} is said to be *State Strictly Dissipative with respect to the supply rate* $w(y,u)$ (or for short SSD$\{w\}$) if there exists a continuously differentiable, positive-definite function $V : \mathbb{R}^n \to \mathbb{R}$, with $V(0) = 0$, called a *storage function*, and a constant $\epsilon > 0$, such that along any trajectory of the system the *dissipative inequality*

$$\dot{V}(x(t)) \leq -\epsilon V(x(t)) + w(y(t), u(t)) \tag{3}$$

is satisfied.

We will consider only quadratic *supply rates* (see [31, 32]):

$$w(y,u) = \begin{bmatrix} y \\ u \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} \tag{4}$$

where $Q \in \mathbb{R}^{p\times p}$, $S \in \mathbb{R}^{p\times m}$, $R \in \mathbb{R}^{m\times m}$ and Q, R are symmetric. In this case the system is said to be (Q,S,R)-SSD. In the case of LTI systems in (2) with quadratic *supply rates* in (4) there is no loss of generality if the *storage function* is restricted to be a positive-definite quadratic form

$$V = x^T P x, \quad P = P^T > 0. \tag{5}$$

This can be characterized by means of a LMI:

Lemma 4

The system Σ_L in (2) is *state strictly dissipative* (SSD) with respect to the supply rate $w(y,u)$ in (4), or for short *(Q,S,R)*-SSD, iff there exist a matrix $P = P^T > 0$ and a constant $\epsilon > 0$ such that

$$\begin{bmatrix} PA + A^T P + \epsilon P & PB \\ B^T P & 0 \end{bmatrix} - \begin{bmatrix} C^T Q C & C^T S \\ S^T C & R \end{bmatrix} \leq 0 \; . \tag{6}$$

A *time-varing nonlinearity* $f : [0,\infty) \times \mathbb{R}^m \to \mathbb{R}^p$ (see [15, 21])

$$y = f(t,u) \tag{7}$$

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piece-wise continuous in t and locally Lipschitz in u, such that $f(t,0)=0$, is said to be *dissipative* with respect to the supply rate $w(y,u)$ in (4), or for short *(Q,S,R)*-D, if for every $t\geq 0$ and $u\in\mathbb{R}^m$,

$$w(y,u)=w(f(t,u),u)\geq 0\,. \tag{8}$$

We note that the classical sector conditions [15] for square nonlinearities, i.e., $m=p$, can be represented in this form:

1. If $f\in[K_1,K_2]$, i.e., $(y-K_1u)^T(K_2u-y)\geq 0$, then it is *(Q,S,R)*-D with: $(Q,S,R)=\left(-I,\frac{1}{2}(K_1+K_2),-\frac{1}{2}\left(K_1^TK_2+K_2^TK_1\right)\right)$.
2. If $f\in[K_1,\infty]$, i.e., $(y-K_1u)^T u\geq 0$, then it is *(Q,S,R)*-D with: $(Q,S,R)=\left(0,\frac{1}{2}I,-\frac{1}{2}\left(K_1+K_1^T\right)\right)$.

A multivariable nonlinearity f can be (Q_i,S_i,R_i)-D for several triples (Q_i,S_i,R_i), i.e., $\omega_i(f(t,u),u)=f^TQ_if+2f^TS_iu+u^TR_iu\geq 0$, for $i=1,2,...,\mu$ [21]. In this case, it is easy to see that f is $\sum_{i=1}^{\mu}\theta_i(Q_i,S_i,R_i)$-D for every $\theta_i\geq 0$, i.e., f is dissipative with respect to the supply rate $\omega_\theta(f,u)=\sum_{i=1}^{\mu}\theta_i\omega_i(f,u)$. Note that the inequality (8) provides a characterization of the nonlinearity f, since it means that the graph of f is contained in the subspace corresponding to the non negative eigenvalues of the quadratic form (Q,S,R). If the quadratic form (Q,S,R) is positive semidefinite, then inequality (8) does not provide any information about the nonlinearity f and it is therefore of no use in what follows.

The following lemma, which is a generalization of the circle criterion of absolute stability for non square systems, shows that the negative feedback interconnection of a LTI dissipative system with a (complementary) dissipative static nonlinearity is internally exponentially stable:

Lemma 5 ([21, 22])
Consider the feedback interconnection

$$\Xi_S:\begin{cases}\dot{x}=Ax+Bu, \quad x(0)=x_0\\ y=Cx\\ u=-f(t,y)\ .\end{cases} \tag{9}$$

If there exist *(Q, S, R)* such that $f(t,y)$ is (Q,S,R)-D, and the linear subsystem of Ξ_S is $\left(-R,S^T,-Q\right)$-SSD, then the equilibrium point $x=0$ of Ξ_S *is globally exponentially stable*, , i.e. there exist constants $k>0$ and $\varrho>0$ such that for all x_0

$$\|x(t)\|\leq k\,\|x_0\|\exp(-\varrho t)\,. \qquad \square \tag{10}$$

Now we consider system Ξ_S with external additive perturbations, i.e.

$$\Xi_P:\begin{cases}\dot{x}=Ax+Bu+b(t), \quad x(0)=x_0\\ y=Cx\\ u=-f(t,y)\end{cases} \tag{11}$$

where $b(t)$ is an input signal. Trajectories of the system will not converge to the origin $x=0$, but if the input signal $b(t)$ is bounded, the trajectories will converge to a ball centered at the origin, with a radius depending on the bound of b, and they will remain there for all future times. This is the content of the *Input-to-State-Stability* property:

Definition 6 ([15])
System Ξ_P is said to be *Input-to-State Stable* (ISS) with respect to $b\left(t\right)$ if there exist a class $\mathcal{KL}$ function β, a class $\mathcal{K}$ function γ such that for any initial state $x(t_0)$ and any bounded input $b(t)$, the solution $x(t)$ exists for all future times $t \geq t_0$ and satisfies

$$\left\|x\left(t\right)\right\| \leq \beta(\left\|x(t_0)\right\|, t-t_0) + \gamma\left(\sup_{t_0 \leq \tau \leq t} \|b(\tau)\|\right)$$

Lemma 7
Consider the system Ξ_P and suppose that the conditions of Lemma 5 are satisfied. Under these conditions system Ξ_P is *ISS with respect to* b. □

Proof
We give the proofs of Lemmata 5 and 7. By hypothesis (4) is satisfied with $\left(-R, S^T, -Q\right)$. Take $V\left(x\right) = x^T P x$ as Lyapunov-like function candidate for the closed loop system. The time derivative of $V\left(x\right)$ along the solutions of Ξ_P is $\dot{V} = \left(Ax + Bu\right)^T Px + x^T P\left(Ax + Bu\right) + 2x^T Pb(t)$, or, because of (6) and (11)

$$\begin{aligned}
\dot{V} &= \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} PA + A^T P & PB \\ B^T P & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + 2x^T Pb(t) \\
&\leq \begin{bmatrix} x \\ -f \end{bmatrix}^T \begin{bmatrix} -C^T RC & C^T S^T \\ SC & -Q \end{bmatrix} \begin{bmatrix} x \\ -f \end{bmatrix} - \epsilon x^T Px + 2x^T Pb(t) \\
&= -\begin{bmatrix} f \\ y \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} f \\ y \end{bmatrix} - \epsilon V\left(x\right) + 2x^T Pb(t) \leq -\epsilon V\left(x\right) + 2x^T Pb(t)\,,
\end{aligned}$$

since f is (Q, S, R)-dissipative. When the perturbation vanishes, i.e. $b(t) = 0$, then $\dot{V} \leq -\epsilon V$ and, using the comparison Lemma [15], it follows that

$$V\left(x\left(t\right)\right) \leq V\left(x(0)\right) e^{-\epsilon t}\,.$$

This means that

$$\lambda_{\min}\left(P\right) \|x(t)\|_2^2 \leq x^T(t) P x(t) \leq x^T(0) P x(0) e^{-\epsilon t} \leq \lambda_{\max}\left(P\right) \|x(0)\|_2^2 e^{-\epsilon t}\,,$$

where $\lambda_{\min,\max}(P)$ are the smallest and the greatest eigenvalues of P, respectively. Therefore

$$\|x(t)\|_2 \leq \sqrt{\frac{\lambda_{\max}\left(P\right)}{\lambda_{\min}\left(P\right)}} \|x(0)\|_2 e^{-\frac{\epsilon}{2}t}\,,$$

so that $x = 0$ is a globally exponential equilibrium point. This finishes the proof of Lemma 5.

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When b is different from zero the inequality $\dot{V} \leq -\epsilon V + 2x^T P b$ can be rewritten, for any $\theta \in (0,1)$, as

$$\begin{aligned}\dot{V}(x) &\leq -(1-\theta)\epsilon V - \theta\epsilon x^T P x + 2x^T P b \\ &\leq -(1-\theta)\epsilon V - \theta\epsilon\lambda_{\max}(P)\|x\|_2^2 + 2\lambda_{\max}(P)\|b\|_2\|x\|_2 \\ &\leq -(1-\theta)\epsilon V + \lambda_{\max}(P)\|x\|_2(2\|b\|_2 - \theta\epsilon\|x\|_2) \\ &\leq -(1-\theta)\epsilon V, \quad \forall \|x\|_2 \geq \frac{2}{\theta\epsilon}\|b\|_2 \,.\end{aligned}$$

Applying [15, Theorem 4.19] it follows that Ξ_P is ISS with respect to $b(t)$. □

3. PRESERVING ORDER OBSERVERS

For a *preserving order observer* the estimated state $\hat{x}(t)$ is always above (or below) the true state variable $x(t)$ of the plant, i.e. either $\hat{x}(t) \succeq x(t)$ or $\hat{x}(t) \preceq x(t)$.

Definition 8 (Preserving Order Observer)
Consider a nonlinear system

$$\Sigma_{\text{NLP}} \,:\, \dot{x} = F(t,x,u,w) \,, \quad y = H(t,x,u) \,, \quad x(0) = x_0 \,,$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ is a known *input*, $y \in \mathbb{R}^p$ and $w \in \mathbb{R}^q$ is an unknown input, representing uncertainties and/or perturbations acting on the system. F and H are smooth and we will assume that the system is *complete*. The dynamical system

$$\Omega_{\text{NLP}} \,:\, \dot{\widehat{x}} = \Phi(t,\widehat{x},u,y,\bar{w}) \,, \quad \widehat{x}(0) = \widehat{x}_0 \,,$$

is an *Upper (Lower) Preserving Order Observer*, for system Σ_{NLP} if (i) it is complete. (ii) The estimation error $e(t) = \widehat{x}(t) - x(t)$ converges globally and asymptotically to zero when the perturbation is identically zero, i.e. $w(t) = 0$. (iii) Whenever the initial states of the observer are greater (smaller) than the initial states of the plant, i.e. $\widehat{x}_0 \succeq x_0$ ($\widehat{x}_0 \preceq x_0$), the estimated state will be greater (smaller) than the state of the plant for all future times and inputs (u,w), i.e. $\widehat{x}(t,t_0,\widehat{x}_0,u,y(t)) \succeq x(t,t_0,x_0,u,w(t))$ or ($\widehat{x}(t,t_0,\widehat{x}_0,u,y(t)) \preceq x(t,t_0,x_0,u,w(t))$). □

Although the completeness condition is not strictly necessary, it simplifies the presentation and it also seems reasonable for physical systems. An *interval observer* [17, 18] can be obtained with two preserving order observers: one upper and the other lower bounding the states of the plant. If we dispense of the convergence condition (ii), then a *framer* will be obtained.

Condition (ii) in Definition 8 is a convergence condition on the observer, while condition (iii) is a cooperativity condition on the estimation error. In order to guarantee the convergence of the observer we will make use of the the dissipativity theory. This idea has been already proposed by one of the authors [21, 22, 27] to design observers for a class of nonlinear systems. For the same class of systems we extend the method in order to make the observer not only convergent, but also order preserving [4]. We consider first the case of systems without perturbations and then a modification

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will be introduced to the observer in order to assure the order preserving property despite of the perturbation.

3.1. Preserving Order Observers for nonlinear systems: The nominal case

Consider the nonlinear plant (without uncertainties/perturbations)

$$\Pi_S : \begin{cases} \dot{x} = Ax + Gf(\sigma; t, y, u) + \varphi(t, y, u), \\ \sigma = Hx, \quad x(0) = x_0 \\ y = Cx \end{cases} \tag{12}$$

where $x \in \mathbb{R}^n$ is the state, $y \in \mathbb{R}^q$ is the measured output, $\sigma \in \mathbb{R}^r$ is a (not necessarily measured) linear function of the state, $u \in \mathbb{R}^p$ is the input, $f(\sigma; t, y, u) \in \mathbb{R}^m$ is a nonlinear function locally Lipschitz in σ, depending on t and the measured variables u and y, and φ is a nonlinear function locally Lipschitz in $(u,\ y)$ and piecewise continuous in t. We propose here a full order observer [21, 22, 27] for Π_S in (12) of the form

$$\Pi_O : \begin{cases} \dot{\widehat{x}} & = A\widehat{x} + L(\widehat{y} - y) + Gf(\widehat{\sigma} + N(\widehat{y} - y); t, y, u) + \varphi(t, y, u), \\ \widehat{\sigma} & = H\widehat{x}, \quad \widehat{x}(0) = \widehat{x}_0 \\ \widehat{y} & = C\widehat{x} \end{cases} \tag{13}$$

where $\widehat{x} \in \mathbb{R}^n$ is the estimate of x in (12) and the matrices $L \in \mathbb{R}^{n\times q}$ and $N \in \mathbb{R}^{r\times q}$ have to be designed. The state estimation error is defined by $e \triangleq \widehat{x} - x$, the output estimation error by $\widetilde{y} \triangleq \widehat{y} - y$, and the functional estimation error by $\widetilde{\sigma} \triangleq \widehat{\sigma} - \sigma$. The dynamics of the observation error are given by

$$\begin{aligned} \dot{e} &= (A + LC)e + G[f(\widehat{\sigma} + N(\widehat{y} - y); t, y, u) - f(\sigma; t, y, u)] \\ \widetilde{y} &= Ce \\ \widetilde{\sigma} &= He \end{aligned} \tag{14}$$

with $e(0) = e_0 = \widehat{x}_0 - x_0$. Note that $\widehat{\sigma} + N(\widehat{y} - y) = Hx + He + NCe = \sigma + (H + NC)e$. Defining $z \triangleq (H + NC)e = \widetilde{\sigma} + N\widetilde{y}$, a function of the estimation error e, and the incremental nonlinearity

$$\phi(z, \sigma; t, y, u) \triangleq f(\sigma; t, y, u) - f(\sigma + z; t, y, u) \tag{15}$$

the dynamics of the error in (14) can be written as

$$\Pi_E : \begin{cases} \dot{e} = A_L e + Gv, \quad e(0) = e_0 \\ z = H_N e \\ v = -\phi(z, \sigma; t, y, u)\ , \end{cases} \tag{16}$$

where $A_L \triangleq A + LC$ and $H_N \triangleq H + NC$.

The observer Π_O in (13) is an (upper/lower) *preserving order observer* for Π_S if the estimation error dynamics Π_E satisfies two properties:

1. The origin $e = 0$ is a globally asymptotically stable equilibrium point, and
2. The error system Π_E in (16) is cooperative (in the classical sense).

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The first condition means that $\widehat{x}(t) \to x(t)$ as $t \to \infty$. The last condition implies that

$$\text{if } \widehat{x}_0 \succeq x_0 \Longrightarrow \widehat{x}(t) \succeq x(t) \ , \ \forall t \geq 0$$
$$\text{if } \widehat{x}_0 \preceq x_0 \Longrightarrow \widehat{x}(t) \preceq x(t) \ , \ \forall t \geq 0 \, ,$$

i.e. the observer preserves the order, and the estimation is always above or below the true plant trajectory, depending on the initial condition. Note that the observer is upper and lower preserving order. In the next paragraphs conditions ensuring these two properties will be given.

3.1.1. Convergence of the observer. The following theorem gives sufficient conditions for asymptotic stability of the origin of the error dynamics making use of the dissipativity theory:

Theorem 9 ([21, 22, 27])
Suppose that the nonlinearity ϕ in (15) is (Q_i, S_i, R_i)-D for some finite set of (non positive semidefinite) quadratic forms

$$\omega_i(\phi, z) = \phi^T Q_i \phi + 2\phi^T S_i z + z^T R_i z \geq 0, \text{ for all } \sigma, \text{ for } i = 1, 2, ..., \mu \tag{17}$$

Suppose that there exist matrices L, N and a vector $\theta = (\theta_1, ..., \theta\mu)$, $\theta_i \geq 0$, such that Π_E in (16) is $\left(-R_\theta, S_\theta^T, -Q_\theta\right)$-SSD with $(Q_\theta, S_\theta, R_\theta) = \sum_{i=1}^{\mu} \theta_i (Q_i, S_i, R_i)$, that is, there exist matrices $P = P^T > 0$, L and N, a vector $\theta = (\theta_1, ..., \theta\mu) \succeq 0$ and $\epsilon > 0$ such that the matrix inequality (MI)

$$\begin{bmatrix} PA_L + A_L^T P + \epsilon P + H_N^T R_\theta H_N & PG - H_N^T S_\theta^T \\ G^T P - S_\theta H_N & Q_\theta \end{bmatrix} \leq 0 \tag{18}$$

is satisfied. Under these conditions the *observer* Π_O in (13) is *globally exponentially stable* for Π_S.
□

The proof of the Theorem is based on Lemma 5 (see the proof of Lemma 7). The Dissipative Design of observers presented in the preceding Theorem 9 is very flexible in the kind of nonlinearities considered for the system. Moreover, the inclusion of multiple quadratic forms (Q_i, S_i, R_i) to characterize this nonlinearity greatly enhances the possibilities of the method. In particular, several methods for observer design, well known in the literature, are special cases of this design. As examples we mention the High-Gain Observer design [10], the circle criterion design [3], the Lipschitz observer design [24], *et cetera*. For more details we refer the reader to the references [21, 22, 27].

3.1.2. Cooperativity of the observation error. The observer in (13) designed according to Theorem 9 is convergent, but it does not have any order preserving properties. In order to assure cooperativity, the error system Π_E in (16) has to be a cooperative system. In accordance to the characterization given in Proposition 2 this will be the case if the Jacobian matrix of the vector field of Π_E is Metzler, i.e.

$$\frac{\partial}{\partial e}\{A_L e - G\phi(z, \sigma; t, y, u)\} = A_L + G\frac{\partial f(z+\sigma; t, y, u)}{\partial z} H_N \overset{M}{\succeq} 0, \ \forall z, \sigma, t, y, u \, .$$

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We obtain this last expression noting that from (15) and the equality $z = H_N e$ it follows that

$$-\frac{\partial \phi\left(z, \sigma; t, y, u\right)}{\partial e} = \frac{\partial f\left(z+\sigma; t, y, u\right)}{\partial e} = \frac{\partial f\left(z+\sigma; t, y, u\right)}{\partial z} H_N \, .$$

Note that this condition is equivalent to the fact that the matrix

$$M\left(z; t, y, u\right) \triangleq A_L + GJ\left(z; t, y, u\right) H_N \overset{M}{\succeq} 0 \, , \; \forall z \in \mathbb{R}^r, \; \forall t, y, u \tag{19}$$

is Metzler everywhere. Here $J(z; t, y, u) = \frac{\partial f(z;t,y,u)}{\partial z}$ is the Jacobian matrix of the nonlinearity of the plant Π_S in (12). The following Theorem provides sufficient conditions for the design of a preserving order observer for Π_S in absence of uncertainties/perturbations:

Theorem 10
Consider the plant Π_S in (12), and assume that the nonlinearity ϕ in (15) is (Q_i, S_i, R_i)-Dissipative for a finite number of quadratic forms, that is (17) holds. Suppose furthermore that there exist constant matrices $P = P^T > 0, \; L, \; N$, a constant scalar $\epsilon > 0$, and a vector $\theta = (\theta_1, ..., \theta_\mu), \; \theta_i \geq 0$, such that:

1. The Dissipativity condition, that is the matrix inequality in (18) is satisfied, and
2. $M\left(z; t, y, u\right) \overset{M}{\succeq} 0$ in (19) is Metzler $\forall z \in \mathbb{R}^r, \; \forall t, y, u$.

Under these conditions the system Π_0 in (13) is a globally exponentially convergent *upper/lower preserving order observer* for the plant Π_S. □

Proof
The Convergence condition 1 is derived from Theorem 9 (see also the Proof of Lemmata 5 and 7). The cooperativity condition 2 follows from Proposition 2. □

Note that the same system Π_0 is an upper or a lower preserving order observer, depending if the initial condition of the observer is above ($\widehat{x}_0 \succeq x_0$) or below ($\widehat{x}_0 \preceq x_0$) the initial conditions of the plant, respectively. Since the initial condition of the plant is unknown, to initialize correctly the upper (lower) preserving order observer it is necessary to know an upper x_0^+ (lower x_0^-) bound of the possible initial conditions of the plant, i.e.

$$x_0^- \preceq x_0 \preceq x_0^+ \, . \tag{20}$$

Selecting the initial condition of the upper (lower) preserving order observer as $\widehat{x}_0 \succeq x_0^+$ ($\widehat{x}_0 \preceq x_0^-$) ensures the correct behavior.

In order to obtain simultaneously a lower and an upper estimate of the state trajectory, it is necessary to built two preserving order observers and initialize them adequately. This constitutes an *interval observer*.

Note that the design of a preserving order observer imposes an additional cooperativity condition to the usual one of convergence for a standard observer. As a consequence it is expected that: (i) The class of systems for which a preserving order observer can be designed is a proper subset of the ones for which a convergent observer exist. (ii) The assignable dynamic properties of a preserving order observer can be more restricted than those for a simple convergent observer. For example, it is

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possible that for a system is possible to design a convergent observer with assignable convergence dynamics, but that the convergence dynamics of a preserving order observer (if it exists) are strongly restricted. The study of these issues is an important topic for further research. Note that to design a preserving order observer it is not necessary that the plant Π_S be cooperative. Finally, the results of the paper can be easily extended to the class of systems described by Π_S in (12) with system matrices depending on time or measurable signals y and u as long as the dissipative inequality is satisfied with *constant* values for P and ϵ. The observer matrices L and N can also be time-varying.

3.2. *Preserving Order Observers for systems with uncertainties/perturbations*

Consider now the plant in (12) on which additive perturbations are acting

$$\Psi_S : \begin{cases} \dot{x} = Ax(t) + Gf(\sigma; t, y, u) + \varphi(t, y, u) + \pi(t, x) \ , \\ \sigma = Hx(t) \, , \quad x(0) = x_0 \\ y = Cx(t) \ , \end{cases} \tag{21}$$

where $\pi(t, x) \in \mathbb{R}^n$ represents unknown exogenous variables or/and system uncertainties. If the observer Π_0 in (13) is used for the plant, then the resulting estimation error dynamics

$$\Pi_E^p : \begin{cases} \dot{e} = A_L e + Gv - \pi(t, x) \ , \quad e(0) = e_0 \\ z = H_N e \\ v = -\phi(z, \sigma; t, y, u) \ , \end{cases}$$

will not converge to zero, if the perturbation is not vanishing. Moreover, since in general $\pi(t, x)$ is not an ordered input signal, the error dynamics will not be cooperative, and the estimated states will not be ordered with respect to the true states of the plant. Due to the presence of unknown inputs in the plant, it is unlikely to obtain an observer that provides exact estimation of the states. In general, we will then require the estimation error not to be convergent, but to be bounded when the uncertainty is bounded, and to converge when the perturbation vanishes. In order to obtain an upper (lower) preserving order observer it will be assumed that upper (lower) bounds for the perturbation $\pi(t, x)$ are known, so that $\pi(t, x)$ satisfies

$$\pi^+(t, y) \succeq \pi(t, x) \succeq \pi^-(t, y) \ , \forall t \geq 0, \ \forall x, y \ , \tag{22}$$

where $\pi^+(t, y)$ and $\pi^-(t, y)$ are known Lipschitz continuous functions.

Consider an upper Ψ_{O^+} (lower Ψ_{O^-}) estimator for Ψ_S of the form

$$\Psi_{O^+} : \begin{cases} \dot{\widehat{x}}^+ = A\widehat{x}^+ + Gf(\widehat{\sigma}^+ + N^+(\widehat{y}^+ - y)) + L^+(\widehat{y}^+ - y) + \pi^+(t, y) + \varphi(t, y, u) \\ \widehat{\sigma} = H\widehat{x}^+, \quad \widehat{x}^+(0) = \widehat{x}_0^+ \\ \hat{y} = C\widehat{x}^+ \end{cases} \tag{23}$$

$$\Psi_{O^-} : \begin{cases} \dot{\widehat{x}}^- = A\widehat{x}^- + G\ f(\widehat{\sigma}^- + N^-(\widehat{y}^- - y)) + L^-(\widehat{y}^- - y) + \pi^-(t, y) + \varphi(t, y, u) \\ \widehat{\sigma} = H\widehat{x}^-, \quad \widehat{x}^-(0) = \widehat{x}_0^- \\ \widehat{y} = C\widehat{x}^- \end{cases} \tag{24}$$

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where $\widehat{x}^+$ ($\widehat{x}^-$) is the upper (lower) estimate of x. Matrices $L^+ \in \mathbb{R}^{n\times q}$, $N^+ \in \mathbb{R}^{r\times q}$ ($L^- \in \mathbb{R}^{n\times q}$, $N^- \in \mathbb{R}^{r\times q}$) have to be designed in order to assure the required convergence and cooperative properties. Note that in Ψ_{O^+} the upper bound and in Ψ_{O^-} the lower bound of the perturbation (22) have been introduced. This will allow the error dynamics to be cooperative. Defining the estimation errors as $e^+ \triangleq \widehat{x}^+ - x$ and $e^- \triangleq x - \widehat{x}^-$ (and the other variables defined in a similar manner), the observation error dynamics are given by

$$\Psi_{E^+} \begin{cases} \dot{e}^+ = A_L^+ e^+ + G v^+ + b^+ \\ z^+ = H_N^+ e^+, \quad e^+(0) = e_0^+ \succeq 0 \\ v^+ = -\phi^+\left(z^+, \sigma\right) \end{cases} \tag{25}$$

$$\Psi_{E^-} \begin{cases} \dot{e}^- = A_L^- e^- + G v^- + b^- \\ z^- = H_N^- e^-, \quad e^-(0) = e_0^- \succeq 0 \\ v^- = -\phi^-\left(z^-, \sigma\right) \end{cases} \tag{26}$$

where $A_L^+ \triangleq A + L^+C$, $H_N^+ \triangleq H + N^+C$, $A_L^- \triangleq A + L^-C$, $H_N^- \triangleq H + N^-C$ and the incremental nonlinearities $\phi^+\left(z^+, \sigma\right)$ y $\phi^-\left(z^-, \sigma\right)$ are defined in a similar form as (15). The uncertainty errors

$$b^+ \triangleq \pi^+\left(t, y\right) - \pi\left(t, x\right) \succeq 0 \tag{27}$$

$$b^- \triangleq \pi\left(t, x\right) - \pi^-\left(t, y\right) \succeq 0 \tag{28}$$

act as inputs on the systems Ψ_{E^+} in (25) and Ψ_{E^-} in (26).

Note that the observer Ψ_{O^+} in (23) (Ψ_{O^-} in (24)) is an upper (lower) *preserving order observer* for Ψ_S if the estimation error dynamics Ψ_{E^+} (Ψ_{E^-}) satisfies two properties:

1. It is ISS with respect to b^+ (b^-), and
2. The error system Ψ_{E^+} (Ψ_{E^-}) is cooperative with input b^+ (b^-) and output e^+ (e^-).

The first condition means that the estimation error $e^+(t)$ ($e^-(t)$) will be ultimately bounded by a bound depending on b^+ (b^-) and that it will converge to zero if $b^+(t) \to 0$ ($b^-(t) \to 0$) as $t \to \infty$. The latter condition implies that

$$\widehat{x}_0^+ \succeq x_0 \Longrightarrow \widehat{x}^+\left(t\right) \succeq x\left(t\right) \;, \left(x_0 \succeq \widehat{x}_0^- \implies x\left(t\right) \succeq \widehat{x}^-\left(t\right)\right) \;, \; \forall t \geq 0 \;,$$

i.e. the observer preserves the order, and the trajectories of Ψ_{0^+} (Ψ_{0^-}) are always above (below) the true trajectories of the plant, despite of the perturbations.

Note that here, in contrast to the nominal case, the upper (lower) preserving order observer is only one sided ordered. This means that $\widehat{x}_0^+ \preceq x_0 \not\Rightarrow \widehat{x}^+\left(t\right) \preceq x\left(t\right)$ ($\widehat{x}_0^- \succeq x_0 \not\Rightarrow \widehat{x}^-\left(t\right) \succeq x\left(t\right)$).

The following Theorem states sufficient conditions for the design of a preserving order observer. We state the result only for the upper preserving order observer Ψ_{O^+}, but it is also valid for the lower preserving order observer Ψ_{O^-}.

Theorem 11

Consider the perturbed plant Ψ_S in (21), and assume that the nonlinearity ϕ^+ in (15) is (Q_i, S_i, R_i)-Dissipative for a finite number of quadratic forms, that is (17) holds. Suppose furthermore that

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there exist constant matrices $P^+ = P^{+T} > 0,\ L^+,\ N^+$, a constant scalar $\epsilon^+ > 0$, and a vector $\theta^+ = (\theta_1^+, ..., \theta_\mu^+),\ \theta_i^+ \geq 0$, such that:

1. The Dissipativity condition, that is the matrix inequality in (18) is satisfied, and
2. $M\left(z; t, y, u\right) \overset{M}{\succeq} 0$ in (19) is Metzler $\forall z \in \mathbb{R}^r,\ \forall t, y, u$.

Under these conditions the system Ψ_{O^+} in (23) is a globally ISS *upper preserving order observer* for the plant Ψ_S. □

Proof
The Convergence condition 1 is derived from Lemma 7. The cooperativity condition 2 follows from Proposition 2, by noting that the input and the output mappings are for our case the identity, and therefore its Jacobian matrix is non negative. □

Since both ϕ^+ and ϕ^- depend only on the same nonlinearity f it follows that if ϕ^+ is (Q_i, S_i, R_i)-D then so is ϕ^-. This also implies that if the conditions of the Theorem 11 are satisfied for Ψ_{O^+}, they will be also satisfied for Ψ_{O^-} with the same parameter values, and in particular, with the same observer matrices $L^+ = L^-$ and $N^+ = N^-$. That means that if it is possible to design an upper preserving order observer it will be also possible to design a lower preserving order observer, and *vice versa*. Moreover, both observers can have (but they do not have to) the same matrices $L^+ = L^-$ and $N^+ = N^-$.

If both upper Ψ_{0^+} in (23) and lower Ψ_{0^-} in (24) preserving order observers are run in parallel, if there exist known bounds on the initial conditions (20) and on the perturbations (22) of the plant, and the observers are adequately initialized, then an *interval observer* is obtained for the perturbed plant Ψ_S, so that the next conditions hold:

$$\widehat{x}_0^+ \succeq x_0^+ \succeq x_0 \succeq x_0^- \succeq \widehat{x}_0^- \implies \widehat{x}^+\left(t\right) \succeq x\left(t\right) \succeq \widehat{x}^-\left(t\right) . \tag{29}$$

Note that Theorem 11 can be seen as a generalization of Theorem 10. In fact, setting $\pi^+(t, y) \equiv 0$ in the upper Ψ_{0^+} in (23) and $\pi^-(t, y) \equiv 0$ in the lower Ψ_{0^-} in (24) preserving order observers we obtain an interval observer in the nominal case. Moreover, the set of values for the observer matrices L and N that satisfy the conditions of Theorems 11 and 10 are the same, since the conditions in these Theorems do not depend on the perturbation.

4. COMPUTATIONAL ISSUES

The design of preserving order (or interval) observers for the considered class of nonlinear systems, in the nominal case or in presence of uncertainties/perturbations, is reduced to the solution of two matrix inequalities (see Theorems 10 and 11):

(i) Convergence requires to find matrices $L,\ N,\ P = P^T > 0$, a constant $\epsilon > 0$ and a vector $\theta = (\theta_1, ..., \theta_\mu) \succeq 0$ such that the Matrix Inequality (MI) (18) is satisfied.

(ii) Cooperativity requires that matrices L and N are such that the family of matrices $M\left(\cdot\right) \overset{M}{\succeq} 0$ in (19) is Metzler.

We will discuss some issues on the solution of these inequalities.

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4.1. The Dissipativity Inequality (18)

The nature of the MI in (18) has been studied in [21, 22, 27]. Here we recall some results and provide some new ones. In general, inequality (18) is *nonlinear* in the unknowns $(L, N, P, \epsilon, \theta)$. However, under some important conditions, (18) becomes a Linear Matrix Inequality (LMI) feasibility problem, for which a solution can be effectively found by several algorithms in the literature [11, 7] and in Matlab.

1. A first observation is that the product ϵP makes (18) nonlinear in the two unknowns ϵ and P. However, the bilinear term ϵP can be always replaced by the term ϵI, where I is the identity matrix, that is linear in ϵ, without altering the feasibility of the MI in (18). To see this, assume that there exist ϵ and P that satisfy (18). The MI (18) can be rewritten as

$$\begin{bmatrix} PA_L + A_L^T P + \varepsilon I + H_N^T R_\theta H_N & PG - H_N^T S_\theta^T \\ G^T P - S_\theta H_N & Q_\theta \end{bmatrix} \leq \begin{bmatrix} \varepsilon I - \epsilon P & 0 \\ 0 & 0 \end{bmatrix} . \tag{30}$$

   If $\varepsilon \leq \lambda_{\min}(P)\,\epsilon$, then it follows that the left hand matrix in (30) is negative semidefinite. Conversely, a similar argument shows that if the left hand matrix in (30) is negative semidefinite, then there exist an $\epsilon > 0$ such that MI (18) is fulfilled. So both inequalities are in fact equivalent. For the further discussion we will assume that in (18) the term ϵP has been replaced by the term ϵI.
2. Further nonlinear terms in (18) are due to the terms $S_\theta H_N$ and $H_N^T R_\theta H_N$, where bilinear terms in θ and N appear, but also a quadratic term in N. If N is fixed, as for example for the High-Gain Observer design, where $N = 0$ [21, 22], then (18) is an LMI in P, ϵ, PL, θ. Note that, once the values of P and PL have been obtained, then the value of L can be easily calculated from $L = P^{-1}(PL)$, since P is invertible.
3. A further alternative is to fix θ. This is a natural situation when the nonlinearity ϕ is (Q, S, R)-Dissipative only for *one* quadratic form, that is (17) holds with $\mu = 1$. This is always the case if the nonlinearity f in (12) is scalar, i.e. $m = r = 1$, or if it is square (i.e. $m = r$) and the nonlinearities are decoupled, i.e. $f(\sigma) = [f_1(\sigma_1), \ldots, f_m(\sigma_m)]$. In these situations multiple (non trivial) dissipativity conditions for ϕ are impossible [22, 27]. We consider different situations:

   (a) If $R_\theta = 0$, for example if all $R_i = 0$, again (18) is an LMI in P, ϵ, PL, N. In some situations it is possible to transform a problem with $R_\theta \neq 0$ into one with $R_\theta = 0$ by a loop transformation (see [27]).

   (b) If $R_\theta > 0$, and using Schur's complement, one can transform inequality (18) into the equivalent Matrix Inequality

$$\begin{bmatrix} -R_\theta^{-1} & -H_N & 0 \\ -H_N^T & PA_L + A_L^T P + \epsilon I & PG - H_N^T S_\theta^T \\ 0 & G^T P - S_\theta H_N & Q_\theta \end{bmatrix} \leq 0 \ , \tag{31}$$

   that is again an LMI in P, ϵ, PL, N.

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(c) If $R_\theta \leq 0$ a *sufficient* condition for (18) is the Matrix Inequality

$$\begin{bmatrix} PA_L + A_L^T P + \epsilon I & PG - H_N^T S_\theta^T \\ G^T P - S_\theta H_N & Q_\theta \end{bmatrix} \leq 0 \ ,$$

that is an LMI in P, ϵ, PL, N.

In general, a solution to (18) can be found by using specialized software for nonlinear Matrix Inequality Problems, e.g. the software PENNON [16]. Alternatively, one can use an efficient LMI solver, fixing iteratively the matrix N, or the vector θ in cases (3a-3c) above.

4.2. The Cooperativity Inequality (19)

Consider the set $\mathcal{J} \subset \mathbb{R}^{m\times r}$ of all values taken by the Jacobian of the nonlinearity f, i.e.

$$\mathcal{J} = \left\{ \Gamma \in \mathbb{R}^{m\times r} \middle| \, \Gamma = J\left(z;t,y,u\right) = \frac{\partial f\left(z;t,y,u\right)}{\partial z}, \forall z,t,y,u \right\} . \tag{32}$$

Inequality (19) is an LMI in L, N, for every $J \in \mathcal{J}$. However, since $\mathcal{J}$ is a set with an uncountable infinite number of points, (19) corresponds to an *infinite* number of LMIs. This is a difficult task, and it is of interest to reduce the number of LMIs to be checked to a finite number. In the following paragraphs it will be shown that it is possible to find a *finite* number q of points Ω_i in $\mathbb{R}^{m\times r}$ such that if inequality (19) is satisfied at all these points, i.e. $A_L + G\Omega_i H_N \overset{M}{\succeq} 0 \ \forall i = 1, \dots, q$, then it will be satisfied for all $J \in \mathcal{J}$.

The basic idea is very simple. We assume that the set $\mathcal{J}$ is bounded. This is the case if the nonlinearity $f\left(\sigma;t,y,u\right)$ is globally Lipschitz in σ, uniformly in (t,y,u). In this case, it is always possible to find a polytope that contains $\mathcal{J}$ (take for example a hypercube containing $\mathcal{J}$). Due to the convexity of inequality (19) it suffices to check it on the vertices of such polytope to conclude that it is satisfied on the whole set $\mathcal{J}$. Refinement of the polytope, by adding extra vertices while reducing its size, will provide less conservative conditions.

We will show that if the nonlinearity $\phi\left(z;\sigma t,y,u\right) = f\left(\sigma;t,y,u\right) - f\left(z+\sigma;t,y,u\right)$, defined in (15), satisfies a sector condition, i.e. if there exist two constant matrices $K_1 \in \mathbb{R}^{m\times r}$ and $K_2 \in \mathbb{R}^{m\times r}$ such that for all σ, t, y, u and all z the inequality

$$\left[\phi\left(z;\sigma,t,y,u\right) - K_1 z\right]^T \left[K_2 z - \phi\left(z;\sigma,t,y,u\right)\right] \geq 0 \tag{33}$$

is satisfied, then the set of values at which the inequality (19) has to be verified can be considerably reduced. In fact, we will show that under condition (33) the set $\mathcal{J}$ is contained in a *convex* set Υ, and that it is only necessary to verify the inequality (19) on the *boundary* of Υ. This is established in the following proposition.

Proposition 12

Let $z \to f(z;t,y,u) : \mathbb{R}^r \to \mathbb{R}^m$ be a multivariable function, continuous in t, y, u and continuously differentiable in z. Suppose that $\phi\left(z;\sigma,t,y,u\right) = f\left(\sigma;t,y,u\right) - f\left(z+\sigma;t,y,u\right)$ is in the sector $[K_1\, ,\, K_2]$, i.e. for some constant matrices $K_1 \in \mathbb{R}^{m\times r}$ and $K_2 \in \mathbb{R}^{m\times r}$ (33) is fulfilled. If the matrices

$$A_L + G\Delta H_N \overset{M}{\succeq} 0 \tag{34}$$

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are Metzler for all $\Delta \in \Upsilon_f$, defined by

$$\Upsilon_f = \left\{ \Gamma \in \mathbb{R}^{m\times r} \Big| \ \ (\Gamma + K_1)^T (K_2 + \Gamma) = 0 \right\}, \tag{35}$$

then condition $M(z;t,y,u) \overset{M}{\succeq} 0$ in (19) is satisfied $\forall\, z \in \mathbb{R}^r$, and all t, y, u.

Proof

The demonstration is given in three parts:

(i) We first show that when the nonlinearity ϕ belongs to the sector $[K_1,\ K_2]$, then $\mathcal{J} \subset \Upsilon$, where the convex set

$$\Upsilon \triangleq \left\{ \Gamma \in \mathbb{R}^{m\times r} \Big| \, [\Gamma + K_1]^T [K_2 + \Gamma] \leq 0 \right\}. \tag{36}$$

By the mean value theorem (see e.g. [30, p. 51]) one can write

$$\phi(z;\sigma,t,y,u) = -\left(\int_0^1 J\left(\sigma + \theta z; t, y, u\right) d\theta \right) z\,, \tag{37}$$

where the matrix $\Theta\left(z;\sigma,t,y,u\right) \triangleq \int_0^1 J\left(\sigma+\theta z; t,y,u\right) d\theta$ is continuous in all its arguments. The sector condition (33) can therefore be equivalently expressed as

$$-\left(\int_0^1 z^T \left[J\left(\sigma+\theta z; t,y,u\right) + K_1\right]^T \left[K_2 + J\left(\sigma+\theta z;t,y,u\right)\right] z d\theta \right) \geq 0\,. \tag{38}$$

This implies that $\mathcal{J} \subset \Upsilon$. It is obvious that if (34) is satisfied for all $\Delta \in \Upsilon$ inequality (19) follows.

(ii) Now we show that Υ is a convex set. Convexity of Υ follows from the fact that the defining inequality of Υ, i.e. $[\Gamma + K_1]^T [K_2 + \Gamma] \leq 0$, using Shur's Lemma, can be rewritten as

$$\begin{bmatrix} K_1^T K_2 + K_1\Gamma + \Gamma^T K_2 & -\Gamma^T \\ -\Gamma & -I \end{bmatrix} \leq 0\,,$$

that is an LMI in Γ.

(iii) Finally we show that if (34) is satisfied for all $\Delta \in \Upsilon_f$, the boundary of Υ, then it is also satisfied for all $\Delta \in \Upsilon$. For this note that if $A_L + G\Delta_1 H_N \overset{M}{\succeq} 0$ and $A_L + G\Delta_2 H_N \overset{M}{\succeq} 0$ for two matrices $\Delta_1\,,\, \Delta_2 \in \Upsilon$, then it follows that for every scalar $\alpha \in [0\,,\,1]$

$$\alpha\left(A_L + G\Delta_1 H_N\right) + (1-\alpha)\left(A_L + G\Delta_2 H_N\right) = A_L + G\left(\alpha\Delta_1 + (1-\alpha)\,\Delta_2\right) H_N \overset{M}{\succeq} 0\,,$$

showing that the inequality (34) is convex in Δ. This implies that it suffices to check (34) on the boundary of Υ.

$\square$

Remark 13

Recall that ϕ being in the sector $[K_1\,,\,K_2]$ is equivalent to ϕ being $(Q,\,S,\,R)$-Dissipative, where $(Q,S,R) = \left(-I, \frac{1}{2}\left(K_1+K_2\right), -\frac{1}{2}\left(K_1^T K_2 + K_2^T K_1\right)\right)$.

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A more general version of the Proposition, with basically the same proof, can be obtained in the following form. When ϕ is $(Q,\ S,\ R)$-Dissipative, with $Q<0$, then

$$\phi^T Q\phi + 2\phi^T Sz + z^T Rz = \int_0^1 z^T \left\{ J^T\left(\vartheta\right) QJ\left(\vartheta\right) - 2J^T\left(\vartheta\right) S + R \right\} zd\vartheta \geq 0$$

where $J\left(\vartheta\right) = J\left(\sigma + \vartheta z; t, y, u\right)$. It is clear that $\mathcal{J} \subset \Upsilon$, where Υ is the set of matrices $\Gamma \in \mathbb{R}^{m\times r}$ such that

$$\Gamma^T Q\Gamma - 2\Gamma^T S + R = \begin{bmatrix} R - \Gamma^T S - S^T\Gamma & \Gamma^T \\ \Gamma & -Q^{-1} \end{bmatrix} \geq 0\,. \tag{39}$$

Since Υ is convex, it is sufficient to check the inequality (34) on its boundary Υ_f, defined as the set of $\Gamma \in \mathbb{R}^{m\times r}$ such that $\Gamma^T Q\Gamma - 2\Gamma^T S + R = 0$.

Note also that if ϕ satisfies several dissipative conditions $(Q_i,\ S_i,\ R_i)$, for $i = 1,\ldots,\mu$, with $Q_i < 0$, then the previous Proposition is valid for each sector condition with a boundary set Υ_{fi}. To simplify the presentation we will restrict ourselves in what follows to the case of one dissipativity condition, or when the vector θ is fixed.

Proposition 12 reduces substantially the size of the set of matrices at which the cooperativity condition (19) has to be checked. In particular, in the scalar case, i.e. $r = m = 1$, and $\sigma \to f(\sigma; t, y, u) : \mathbb{R} \to \mathbb{R}$, the set Υ is the interval $[K_1,\ K_2]$ and its boundary are the two extreme points $\Upsilon_f = \{K_1,\ K_2\}$. In this case it suffices to check (19) for just these two values of J. A similar situation occurs in the diagonal case, i.e. $r = m$ and $\sigma \to f(\sigma; t, y, u) : \mathbb{R}^m \to \mathbb{R}^m$ with $\sigma_i \to f_i(\sigma_i; t, y, u) : \mathbb{R} \to \mathbb{R}$ for $i = 1,\ldots,m$, where Υ_f consists of a finite number of points.

Beyond these two simple cases, the number of points in Υ_f is not finite. In what follows we will restrict our discussion to the case where Γ is a vector, i.e. $r = 1$, $m > 1$, since in this case the geometrical interpretation is simple. However, the general idea can be extended to the matrix case. The set of vectors $\Gamma \in \mathbb{R}^m$, satisfying expression in (39), i.e.

$$\Upsilon \triangleq \left\{ \Gamma \in \mathbb{R}^m |\, \Gamma^T Q\Gamma - 2\Gamma^T S + R \geq 0 \right\}$$

where $Q < 0$, represents an ellipsoid in $\mathbb{R}^m$ [7] (see Figure 1).

Given an ellipsoid Υ we construct two convex polytopes:

(i) Select $k > 1$ points $\Delta_i \in \Upsilon_f \subset \mathbb{R}^m$ on the boundary Υ_f of Υ, and construct the closed convex polytope $P_I \subset \mathbb{R}^m$ as the convex hull (convex combination) of these points [13], i.e.

$$P_I = \left\{ \sum_{i=1}^{k} \alpha_i \Delta_i \in \mathbb{R}^m \,\middle|\, \alpha_i \geq 0,\ \sum_{i=1}^{k} \alpha_i = 1,\ \ i = 1, ..., k \right\}. \tag{40}$$

Δ_i, $i = 1,\ldots,k$ are the vertices of P_I. In particular, we select all the vertices of the ellipsoid as vertices of the polygon P_I. In a refinement of P_I further points on Υ_f are added to the set of vertices of P_I. It is easy to see that the polytope P_I is inscribed in Υ, i.e. $P_I \subset \Upsilon$ (see Figure 1).

(ii) At each vertex Δ_i of P_I there is a *supporting hyperplane* H_i, that is tangent to Υ_f. The polytope $P_C \subset \mathbb{R}^m$ can be equivalently defined as the intersection of the closed half-spaces obtained from the supporting hyperplanes, or as the convex hull of the points of intersection Ω_i of these supporting hyperplanes H_i (see [13]), i.e.

$$P_C = \left\{ \sum_{i=1}^{\kappa} \alpha_i \Omega_i \in \mathbb{R}^m \,\middle|\, \alpha_i \geq 0,\ \sum_{i=1}^{\kappa} \alpha_i = 1,\ \ i = 1, ..., \kappa \right\}. \tag{41}$$

Ω_i, $i = 1, \ldots, \kappa$ are the vertices of P_C (see Figure 1). It is also easy to show that the polytope P_C circumscribes Υ, i.e. $\Upsilon \subset P_C$.

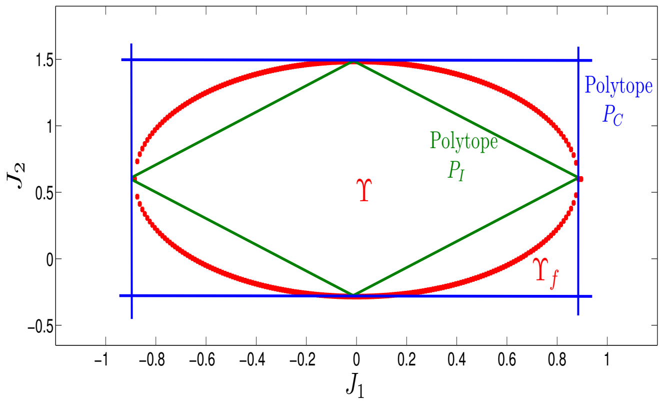

Figure 1. Approximations of the ellipsoid Υ in $\mathbb{R}^2$: P_C is the polytope circumscribing Υ_f and P_I is inscribed in Υ_f.

The vertices of both polytopes P_I and P_C play a crucial role here, since they are a finite number of points (see Figure 1) at which the cooperativity condition is tested. This corresponds to a finite number of LMI's.

To satisfy that the Jacobian matrix $A_L + GJ(z)H_N \overset{M}{\succeq} 0$ in (19) $\forall\, z \in \mathbb{R}^r$ it is necessary that

Coop1) Suppose that the cooperativity condition $A_L + GJH_N \overset{M}{\succeq} 0$ in (19) is satisfied for every $J \in \mathcal{J}$. Then $A_L + G\Delta_i H_N \overset{M}{\succeq} 0$ is satisfied for every vertex $\{\Delta_1, .., \Delta_k\}$ of P_I.

Coop2) Suppose that $A_L + G\Omega_i H_N \overset{M}{\succeq} 0$ is satisfied at each vertex $\{\Omega_1, ..., \Omega_k\}$ of P_C. Then $A_L + GJH_N \overset{M}{\succeq} 0$ in (19) is satisfied for every $J \in \mathcal{J}$.

The proof of these two facts is similar to the proof of the previous Proposition 12. Checking the cooperativity condition on the inscribed polytope P_I constitutes a *necessary* condition for (19). If it is not satisfied at *some* vertex of P_I, then the condition in (19) is falsified. On the other side, checking the cooperativity condition on the circumscribed polytope P_C constitutes a *sufficient* condition for (19). If it is satisfied at *all* vertices of P_C, then the condition in (19) is verified.

The verification of the cooperativity condition (19) can be done iteratively. Suppose that at the step (i) condition **Coop1)** above is satisfied at all vertices of P_I, but condition **Coop2)** is not fulfilled at some vertex Ω_x of P_C. It is reasonable to refine the polytopes by adding some vertices near to Ω_x. The idea is to obtain vertices closer to the boundary Υ_f. A refining procedure is as follows (see Figure 2):

a) Draw a line from the center of Υ to the vertex Ω_x, and find the intersection Δ_{k+1} of this line with Υ_f. Δ_{k+1} is a vertex of the refined polytope P_I.

DOI: 10.1002/rnc

b) Construct the supporting hyperplane (tangent hyperplane) H_{k+1} of Υ at Δ_{k+1} and find the intersection points with all other supporting hyperplanes H_i. These points define new vertices $\Omega_{\kappa+1}, \ldots, \Omega_{\kappa+g}$ for the refined polytope P_C.
c) Conditions **Coop1)** and **Coop2)** are checked at the new vertices of P_I and P_C. If the result is not conclusive, then a new refinement can be performed, i.e. go to item a).

The iteration ends when a conclusive result (falsification or verification of the cooperativity condition) is obtained, or when the number of iterations is sufficiently large.

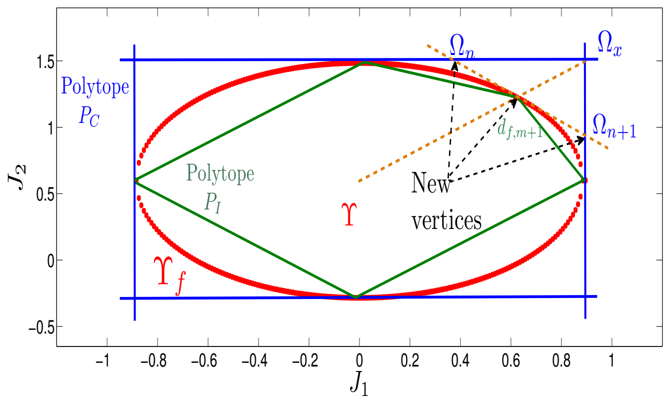

Figure 2. Refinement process: creating new vertices of P_C and P_I.

4.3. *Coupling of the Cooperativity and the Dissipativity Conditions*

In the previous paragraphs we have seen that the cooperativity condition (19) can be reduced to a finite number of LMI's in the design variables L, N. The Dissipativity condition (18), although in general a Nonlinear Matrix Inequality, can be reduced under some important circumstances to a LMI in the design parameters $P, \epsilon, PL, N, \theta$. Note, however, that when trying to solve simultaneously both inequalities a problem appears: whereas (19) is linear in L the inequality (18) is linear in PL, so that both inequalities are coupled in a nonlinear (bilinear) manner.

One possible solution is to solve the inequalities in an iterative form, but there is no guarantee of convergence of this procedure. We propose here a further solution to this problem, consisting in looking at inequality (18) as a Bilinear Matrix Inequality (BMI) in P and L, and using appropriate software for its solution. Thus, the coupling of the variables of both inequalities (18) and (19) is not considered, and therefore, the design of the preserving order observers consists in the solution of a BMI and a LMI. The first one is given by the inequality (18) and the second one by (19).

BMI's are well known in control (see [11, 29] and the references therein), and there are several algorithms to solve it, as e.g. PENBMI [16]. The conditions under which (18) becomes a BMI are the same studied in the first paragraph of this section to render it a LMI. We have implemented the previous solution, including Proposition 12 as well as the refinement criterion, in a numeric algorithm with the help of the software PENBMI and YALMIP. This software has been used for the calculation of the examples presented in the following section.

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5. DESIGN EXAMPLES

To illustrate the design methodology and the performance of the preserving order observers, we will present two examples. The first example shows an experimental validation in the classical three-tanks system. The second example compares in simulation our design with one example taken from the literature.

5.1. Experimental validation: The Three-Tanks system

For the classical Three-Tanks System (we have used Amira's model DTS200) we have designed interval observers using the method proposed in the previous sections, and we have tested the results in simulation and validated them experimentally. This system can be viewed as a prototype of many industrial process applications and our results provide an illustration of the behavior and properties of preserving order observers in a realistic framework (see Figure 3).

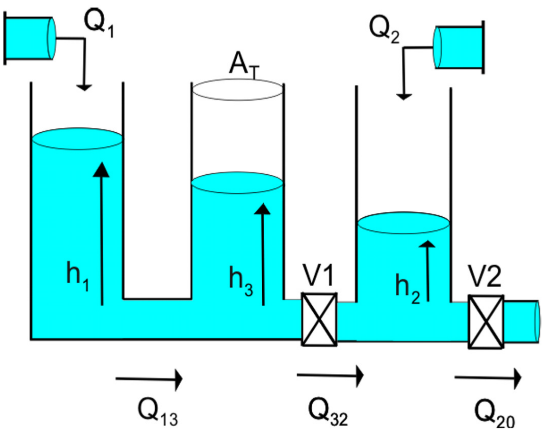

Figure 3. Schematic Diagram of the Three-Tanks System.

The observation problem consists in estimating the level h_3 of the liquid in Tank 3, from the on-line measurements of the levels h_1 and h_2 in the Tanks 2 and 3, respectively. A mathematical description of the behavior of the levels in the three tanks, $[x_1, x_2, x_3] = [h_1,\ h_2,\ h_3]$, which are the states of the system, is given by the equations

$$\Sigma_T : \begin{cases} A_T \frac{dx_1}{dt} = Q_1\,(t) - Q_{13}(x_1 - x_3) \\ A_T \frac{dx_2}{dt} = Q_2\,(t) + Q_{32}(x_3 - x_2) - Q_{20}(x_2) \\ A_T \frac{dx_3}{dt} = Q_{13}(x_1 - x_3) - Q_{32}(x_3 - x_2) \end{cases} . \tag{42}$$

A_T represents the area of the base of the three identical cylindrical tanks. The (volumetric) input flows in Tanks 2 and 1 are Q_2 and Q_1, respectively. Q_{32} is the flow between tanks 2 and 3, that flows through valve $V1$, and Q_{20} is the flow, through valve $V2$, going from Tank 2 outside of the system. Q_{13} is the flow between Tanks 1 and 3. These flows, that are dependent on the difference of levels of the tanks they connect, are described by:

$$
\begin{aligned}
Q_{32}(x_3 - x_2) &\triangleq K_{32}\,\mathrm{sgn}\,(x_3 - x_2)\sqrt{2g\,|x_3 - x_2|}\\
Q_{13}(x_1 - x_3) &\triangleq K_{13}\,(x_1 - x_3)\\
Q_{20}(x_2) &\triangleq K_{20}\sqrt{2gx_2}
\end{aligned}
$$

where $K_{13} = a_{13}S_n$, $K_{32} = a_{32}S_n$ and $K_{20} = a_{20}S_n$. The values of these parameters for the experimental setup used are listed in Table I. It is possible to write the (nominal) system in the

Parameters	Numeric values
S_n	$5\times10^{-5}\ m^2$
g	$9.81\ m/s^2$
a_{13}	8.5720
a_{20}	0.7026
a_{32}	0.4883
$\frac{1}{A_T}$	$64.977\ 1/m^2$

Table I. Parameters for the Experimental setup of the Three-Tanks system Σ_T.

form Π_S in (12) with matrices given by:

$$
A = \frac{K_{31}}{A_T}\begin{bmatrix} -1 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & -1 \end{bmatrix},\ G = \frac{1}{A_T}\begin{bmatrix} 0\\ -1\\ 1 \end{bmatrix},\ H = \begin{bmatrix} 0\\ 1\\ -1 \end{bmatrix}^T,\ C = \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0 \end{bmatrix}.
$$

The known input vector u consists of the two input flows Q_1 and Q_2, i.e. $u = [Q_1\ ,\ Q_2]^T$, and the input/output injection vector $\varphi\,(t, y, u)$, consisting of known signals affecting the dynamics of the plant, is given by

$$
\varphi\,(t, y, u) = \frac{1}{A_T}\begin{bmatrix} Q_1\,(t)\\ Q_2\,(t) - Q_{20}\,(x_2)\\ 0 \end{bmatrix}.
$$

Finally, the (internal) nonlinearity in the model $f\,(\sigma)$ is given by $f\,(\sigma) \triangleq K_{32}\,\mathrm{sgn}\,(\sigma)\sqrt{2g\,|\sigma|}$, and it depends on the variable $\sigma = x_2 - x_3$, which is linear function of the states, i.e. $\sigma = Hx$, and it is not measurable, since it depends on the unmeasured state variable x_3.

We will consider two scenarios for the interval observer design:

1. We consider that there are no uncertainties in the model description, and we design the observer using the results of Subsection 3.1.
2. We consider a perturbation for the system consisting of a leak in Tank 3, flowing out of the system. It is possible to write the (perturbed) system in the form Ψ_S in (21) with the same matrices as for the nominal model and with a perturbation vector $\pi(t, x) = [0,\ 0,\ \pi_3(t)]^T$. We know that $\pi_3(t)$ lies between some values $\pi^- \le \pi(t) \le \pi^+$. We design an interval observer using the results of Subsection 3.2.

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5.1.1. Interval observer design for the Three-Tanks system. For both scenarios we design the same observer matrices L and N. The incremental nonlinearity ϕ defined in (15) is given by

$$\phi(z,\sigma) = K_{32}\left(\operatorname{sgn}(\sigma)\sqrt{2g|\sigma|} - \operatorname{sgn}(\sigma+z)\sqrt{2g|\sigma+z|}\right),$$

which is a scalar nonlinearity in the sector $[-\infty, 0]$. Although it is possible to satisfy the conditions of Theorem 10 using this sector, the required observer gains are very high. Since the infinity value is due to the lack of differentiability of the square root function in Q_{20}, it is reasonable to consider a smooth approximation of this function, that has a finite slope in zero, without altering the accuracy of the model to predict the experimental results. We have found that using an approximating function, belonging to the the sector $\phi \in [-0.1\ ,\ 0]$, or equivalently, being (Q,S,R)-Dissipative with $(Q,S,R) = (-1,\ -0.05,\ 0)$ reasonable experimental results are obtained, with gains implementable in practice. Using this data we solve both, the dissipative in (18) and the cooperative in (19) inequalities using a BMI solver†, and we have obtained as solutions the following matrices

$$L = \begin{bmatrix} -518.317 & 53.9564 \\ 118.9967 & -303.4840 \\ 122.9429 & 278.2979 \end{bmatrix},\ N = \begin{bmatrix} 0 & 10 \end{bmatrix},$$

$$P = \begin{bmatrix} 18.6334 & 39.3332 & 39.3332 \\ 39.3332 & 84.2404 & 84.2334 \\ 39.3332 & 84.2334 & 84.2327 \end{bmatrix},\ \epsilon = 0.1796\ .$$

Since Theorem 10 is satisfied it is possible to design a preserving order observer, and therefore an interval observer by running in parallel two adequately initialized preserving order observers, one to upper estimate and the other to lower estimate the state trajectory. We have used the same gains for both observers, and thus $L = L^+ = L^-$, $N = N^+ = N^-$, $P = P^+ = P^-$, $\epsilon = \epsilon^+ = \epsilon^-$. For comparison purposes an open-loop observer has been designed, i.e.,

$$L = 0 \ \text{ and } \ N = 0.$$

Note that this open loop observer is also a preserving order observer, since the conditions of Theorem 10 are satisfied with these values of L and N.

5.1.2. Experimental validation of the Interval observers. We have run three experiments on the system to test the two interval observers designed. The first two consider the nominal case, while in the third experiment a leak in Tank 3 causes a perturbation for the observation problem.

Experiment 1: For this test we consider the case without a leak in Tank 3. The input flows have been set *constant* with values $Q_1^0 = 2.4425 \times 10^{-5}\,\frac{\mathrm{m}^3}{\mathrm{s}}$ and $Q_2^0 = 1.9889 \times 10^{-5}\,\frac{\mathrm{m}^3}{\mathrm{s}}$. The initial states for the plant were $x_{10} = 0.14$ m, $x_{20} = 0.04$ m and $x_{30} = 0.09$ m, and the initial conditions

† we have used here PENBMI 's software.

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for both the interval and the open loop observers were set as $\widehat{x}_0^+ = (0.22, 0.13, 0.16)$ for the upper, and $\widehat{x}_0^- = (0.07, 0.02, 0.05)$ for the lower preserving order observer.

Figure 4 shows the behavior of the real states and the states estimated by the interval and the open loop observers under these conditions. We first note that both observers are order preserving, so that the estimated states are above/below their real values for all the time. Note further that the estimated states $\widehat{x}_1^+$ and $\widehat{x}_1^-$, $\widehat{x}_2^+$ and $\widehat{x}_2^-$ of the interval observer, that correspond to measured variables, converge very fast, but this is not the case for the open loop observer. The convergence of the estimation for the unmeasured state x_3 is not as fast as for the measured ones, but the estimation errors e_3^+ and e_3^- converge much faster for the interval than for the open loop observer. This shows that the inclusion of the injection (or innovation) terms $N\left(\widehat{y}^{\pm} - y\right)$ and $L\left(\widehat{y}^{\pm} - y\right)$ in the interval observer accelerates the convergence of the estimation. We note that e_3 for the interval observer grows very fast at the initial phase, and then it decreases monotonically. This explains why in Figure 4 the initial states for both observers are apparently different. In this case, since no uncertainties have been considered for the model of the plant, the estimated states of both observers converge (exponentially) to the true values.

Experiment 2: For this test we consider the case without a leak in Tank 3. The input flows are *varying periodically* as $Q_1(t) = 7.3284 \times 10^{-5} \sin(0.1t) \frac{\mathrm{m}^3}{\mathrm{s}}$ and $Q_2(t) = 1.9889 \times 10^{-5} \frac{\mathrm{m}^3}{\mathrm{s}}$. The initial conditions for the plant were $x_{10} = 0.14$ m, $x_{20} = 0.04$ m and $x_{30} = 0.15$ m, while the initial conditions for both observers are the same as in the previous Experiment 1.

Figure 5 shows the behavior of the real state x_3 and its estimation by the interval observer under these conditions. Note again that the estimated states are above/below their real values for all the time, and that they converge (exponentially) to their true values.

Experiment 3: For this test we consider the case *with* a leak in Tank 3. The input flows are kept *constant*, taking the same values as in Experiment 1. The initial conditions for the plant and the interval observer are also as in the previous Experiment 1. The bounds for the leak in the third Tank is considered to be neither bigger than $\pi_3^+ = 3.5899 \times 10^{-6} \frac{\mathrm{m}^3}{\mathrm{s}}$ nor smaller than $\pi_3^- = 1.5763 \times 10^{-6} \frac{\mathrm{m}^3}{\mathrm{s}}$, so that (22) holds.

Figure 6 shows the behavior of the real state x_3 and its estimation by the interval observer under these conditions. Note that the estimated states are above/below their real values for all the time. However, in contrast to the nominal case considered in the two previous Experiments the estimation error does not converge anymore to zero, but it converges (exponentially) to a final bound. The estimated lower bound x_3^- of the (unmeasured) level in Tank 3 can serve as an indicator when the Tank will run empty, before it actually happens. Or the estimated upper bound x_3^- of the same state is an indicator of a possible overflow in the same tank.

5.2. *A Stirred Tank Reactor*

Consider a stirred tank reactor (STR) [26]:

$$\Sigma_{2a} \begin{cases} \dot{x}_b = -D\left(t\right) x_b + \mu\left(s\right) x_b \\ \dot{s} = D\left(t\right)\left(S_{in} - s\right) - \frac{1}{Y}\mu\left(s\right) x_b \\ y = s \end{cases} \tag{43}$$

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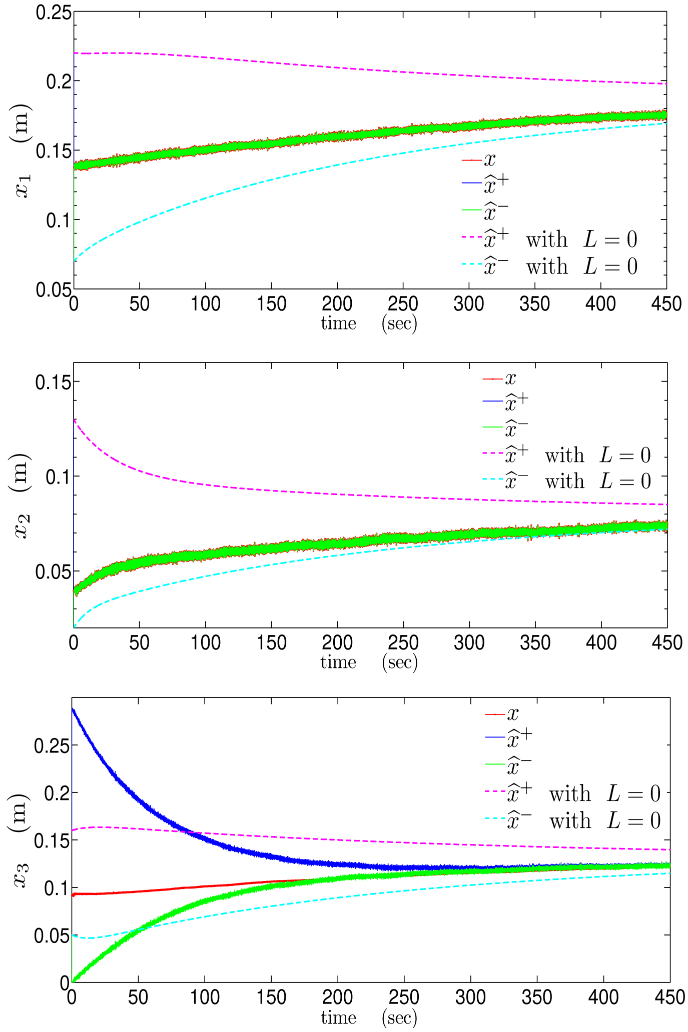

Figure 4. Real and estimated states for the Three-Tanks System with constant inputs in Experiment 1. The real state is a continuous — (red) line, the estimated states of the interval observer are the continuous — (blue and green) lines, and the estimated states of the open loop observer are the dashed – – (magenta and light blue) lines.

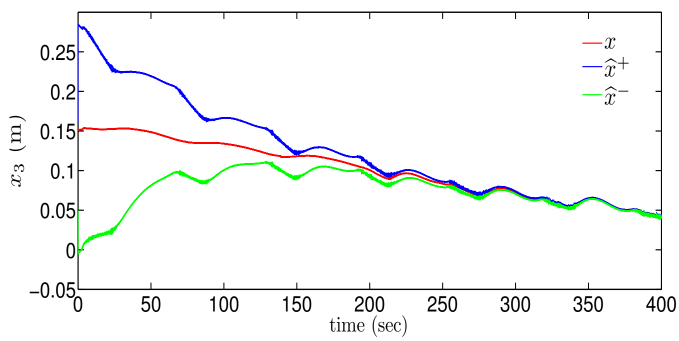

Figure 5. Real and estimated state x_3 for the Three-Tanks System with varying inputs in Experiment 2. The real state is a continuous — (red) line and the estimated states of the interval observer are the continuous — (blue and green) lines

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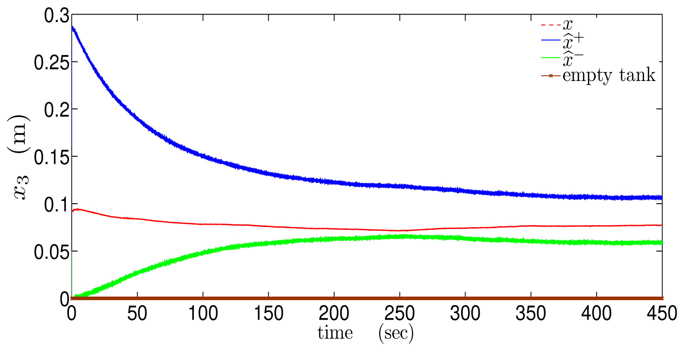

Figure 6. Real and estimated state x_3 for the Three-Tanks System with a leak in Tank 3 in Experiment 3. The real state is a continuous — (red) line and the estimated states of the interval observer are the continuous — (blue and green) lines

where Y is the yield coefficient, μ is the specific growth rate, D is the dilution rate, s is the substrate concentration in the reactor, S_{in} is the influent substrate concentration, and x_b is the biomass concentration in the reactor. The specific growth rate we are going to consider is given by the Monod law

$$\mu(s) = \mu_0 \frac{s}{k_s + s}, \tag{44}$$

where μ_0 is the maximum specific growth rate and k_s is the saturation constant. In [26] interval observers for this system were proposed, with the aim of estimating the biomass concentration x_b from on-line measurements of s, and having as uncertainties the specific growth rate μ and S_{in}. Moreover, a detailed analysis of the convergence conditions and the limits of the achievable behavior of the observers were given.

Here we will illustrate the results using our approach and the relation with the detailed work in [26]. We will consider again two scenarios for the interval observer design: (i) We consider that there are no uncertainties in the model description, and we design the observer using the results of Subsection 3.1. (ii) We consider that the specific growth rate μ is uncertainand we design an interval observer using the results of Subsection 3.2.

5.2.1. Design of an interval observer: The nominal case. For this example it is impossible to design an interval observer in the coordinates of Σ_{2a} since the cooperativity condition (19) cannot be fulfilled, as already noticed in [26]. To explain this, notice first that for this case matrix M has the form

$$M(z; t, y) = \begin{bmatrix} -D(t) + l_1 - \mu(y)\frac{N}{Y} & -\mu(y)\frac{1}{Y} \\ l_2 + \mu(y)N & -D(t) + \mu(y) \end{bmatrix}.$$

Since there is a negative off-diagonal term, that does depend on L, N, the the cooperativity condition is violated by the observer. However, and since the cooperativity is a coordinate dependent property, it is possible to design a preserving order observer in the new coordinates $x = \begin{bmatrix} x_b, & z \end{bmatrix}^T$, where $z = x_b + Ys$, where the model is given by

$$\Sigma_{2b} \begin{cases} \dot{x}_b = -D(t)x_b + \mu(s)x_b \\ \dot{z} = -D(t)z + D(t)YS_{in} \\ y = \frac{1}{Y}(z - x_b). \end{cases} \tag{45}$$

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It is possible to write this (nominal) system in the form Π_S in (12) with matrices given by:

$$A = \begin{bmatrix} -D(t) & 0 \\ 0 & -D(t) \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$H = \begin{bmatrix} 0 & 1 \end{bmatrix}, \ C = \begin{bmatrix} \frac{1}{Y} & -\frac{1}{Y} \end{bmatrix}.$$

The input/output injection vector $\varphi(t, y, u)$, consisting of known signals affecting the dynamics of the plant, is given by

$$\varphi(t, y) = \begin{bmatrix} D(t)\, Y \\ 0 \end{bmatrix} S_{in}(t),$$

and $f(\sigma, y) = \mu(y)\,\sigma$. The incremental nonlinearity $\phi(\sigma, z) = \mu(y)\,\sigma - \mu(y)(\sigma + z) = -\mu(y)\, z$ is $(Q,\ S,\ R)$-D with $(Q,\ S,\ R) = (-1, -0.5, 0)$, i.e. $\phi \in [-1, 0]$. Convergence of the observer is guaranteed if the dissipativity inequality (18) is satisfied. For this case matrix M in (19) is

$$M(z; t, y) = \begin{bmatrix} -D(t) + \frac{l_1}{Y} & -\frac{l_1}{Y} \\ \frac{l_2}{Y} + \mu(y)\frac{N}{Y} & -\left(D(t) + \frac{l_2}{Y}\right) + \mu(y)\left(1 - \frac{N}{Y}\right) \end{bmatrix},$$

so that the Cooperativity Condition is satisfied if $l_1 \leq 0$ and $l_2 + \mu(y)\, N \geq 0$. When both dissipativity (18) and cooperativity (19) inequalities are satisfied, so that Theorem 10 is satisfied, an interval observer of the form (23)-(24) can be designed, with the terms $\pi^+ = \pi^- = 0$. Note that this interval observer is similar to the one proposed by [26], except for two differences: (i) For the shake of the presentation, we have assumed that $\widehat{x}^-$ is always nonnegative, so that our observer is slightly simpler than the one in [26]. Our assumption is reasonable, and it is usually satisfied. (ii) Our interval observer has an extra injection term, $N^{\pm}(\widehat{y}^{\pm} - y)$, entering in the nonlinearity. However, since the nonlinearity in this example is affine in the unmeasured state, i.e. $\mu(s)\, x_b$, then the estimation error is in fact linear time varying, and the injection term due to N becomes also a linear one, adding to the correction term with gain l_2. Therefore, the extra correction term causes no qualitatively different behavior to the observer in [26] for this example. This will be clear in the simulations below. We note, however, that our method generalizes the one proposed in [26].

Using the same parameters as in [26], i.e. $k_s = 5$ g/l, $S_{in} = 5$ g/l, $D = 0.05\ \mathrm{h}^{-1}$, $\mu_0 = 0.33\ \mathrm{h}^{-1}$ and $Y = 0.5$, where D and S_{in} are constant, it is possible to find numerically (using e.g. the PENBMI software) that for the values

$$L = \begin{bmatrix} 0 \\ 19.991 \end{bmatrix}, \ N = -10\,, \quad P = 10^6 \begin{bmatrix} 7.8109 & -0.0007 \\ -0.0007 & 0.00004 \end{bmatrix}, \ \epsilon = 183.232$$

both inequalities (18) and (19) are satisfied. We select in interval observer $N^+ = N^- = N$ and $L^+ = L^- = L$. Figure 7 shows the behavior of the interval observer. The initial conditions for the plant were $x_{b0} = 1.05$ g/l and $s_0 = 0.9$ g/l, and those for the interval observer were set as

$$\begin{aligned} \hat{x}_b^+(0) &= 1.5 \text{ g/l}\,,\ \hat{x}_b^-(0) = 0 \text{ g/l}, \\ \hat{z}^+(0) &= 0.45 \text{ g/l}\,,\ \hat{z}^-(0) = 1.95 \text{ g/l}\,. \end{aligned}$$

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As expected the estimation error preserves the order and it converges exponentially to zero. As explained in [26] the convergence velocity cannot be improved much further without violating the cooperativity condition.

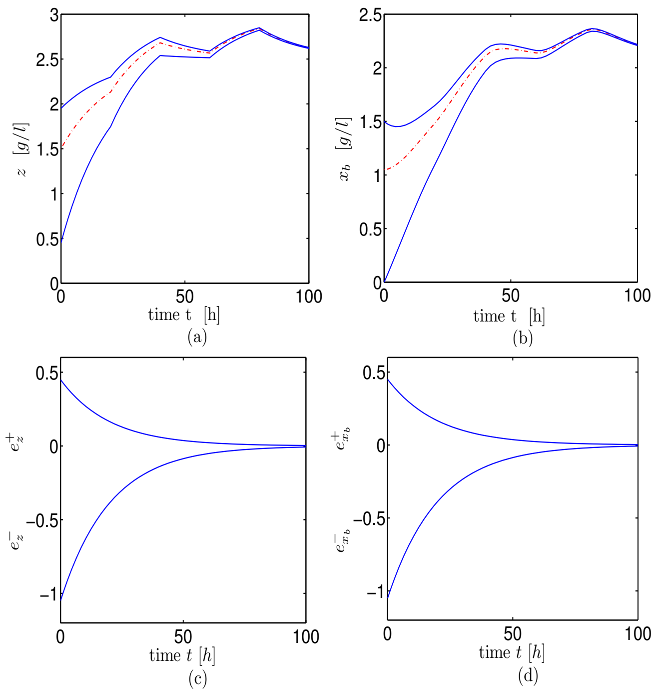

Figure 7. Simulation of the bioreactor (–.), the estimation of the proposed interval observer (–) and their estimation errors for the nominal case.

5.2.2. Interval observer design for the stirred tank reactor with uncertainties We consider now that the specific growth rate μ in (44) is uncertain, and known to be between lower and upper bounds

$$\mu^-(s) \leq \mu(t,s) \leq \mu^+(s) \ .$$

In particular, we assume that $\mu^{\pm}(s)$ are given by Monod models (44) with $\mu_0^+ = 0.495\ \mathrm{h}^{-1}$ and $\mu_0^- = 0.165\ \mathrm{h}^{-1}$. The interval observer is of the form (23)-(24), where the terms π^+ and π^- are given by

$$\pi^+(t,y) = 0.165\frac{y}{k_s+y}x_b^+ \ , \ \pi^-(t,y) = -0.165\frac{y}{k_s+y}x_b^- \ .$$

Using the same parameters as in the previous paragraph we found numerically a different set of parameters satisfying both inequalities (18) and (19)

$$L = \begin{bmatrix} -0.5847 \\ 561.2413 \end{bmatrix} \ , \ N = -10 \ , \ \ P = 10^4 \begin{bmatrix} 4.2090 & -0.0025 \\ -0.0025 & 0.0068 \end{bmatrix} \ , \ \epsilon = 506.6 \ .$$

We also select in interval observer $N^+ = N^- = N$ and $L^+ = L^- = L$. For comparison we design also the interval observer proposed in [26], that corresponds to the structure given by (23)-(24) with $l_1 = 0$, $l_2 = 2.255$, and $N = 0$. Figure 8 shows the behavior of the trajectories for the plant, both

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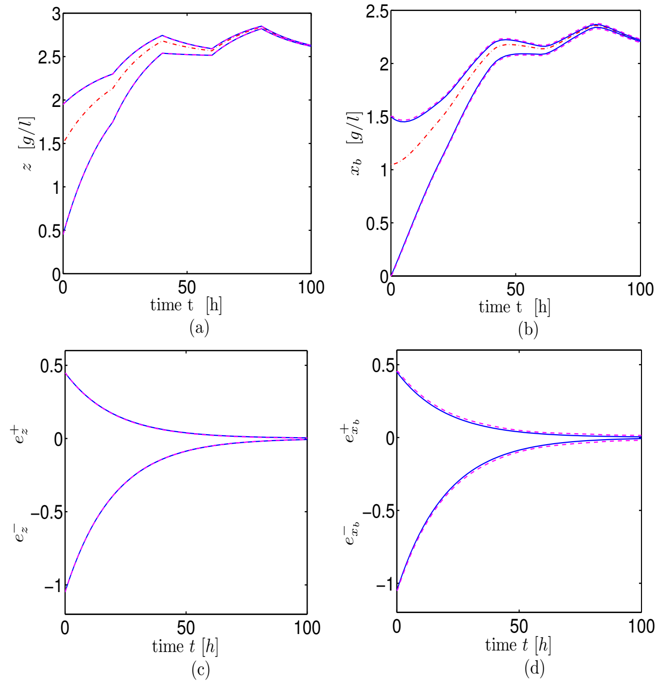

Figure 8. Simulation trajectories of the bioreactor $(-.)$, the upper and lower estimates of our interval observer $(--)$, the interval observer proposed in [26] $(--)$, and their estimation errors when there is an uncertainty in the maximum specific growth rate μ_0.

interval observers and their respective estimation errors when the initial conditions are the same as in the previous paragraph.

As expected the estimation error preserves the order and it converges exponentially to an ultimate bound (different from zero). Again the convergence velocity cannot be improved much further without violating the cooperativity condition. Since both interval observers are essentially identical, they have very much the same behavior. Anew, the interesting point here is that the preserving order observers methodology allows to unify several interval observers design methods.

6. CONCLUSIONS

A novel methodology to design preserving order observers for a class of nonlinear systems in absence and in presence of perturbations/uncertainties has been proposed. This methodology unifies several interval observers design methods proposed previously in the literature and it is based on the combination of dissipative and cooperative properties, both applied to the estimation error dynamics. Computationally the design can be reduced, in most cases, to the solution either of a Bilinear Matrix Inequality (BMI) and a Linear Matrix Inequality (LMI), or of two coupled LMIs, for which highly efficient numerical methods are available. The design methodology has been validated experimentally in the three-tanks system. It would be interesting in future work to analyze the effect of measurement noise on the estimation of the preserving order observers (a work already initiated in [26]) and to use state transformations in a more systematic way to design interval observers, in the same spirit as the recent works of [17, 18].

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