Coupled oscillators (Normal-modes approach) *

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Abstract

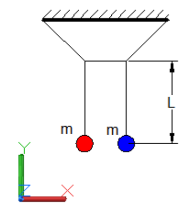
Many important physics systems involved coupled oscillators. A solid is a good example because it can be described in terms of coupled oscillations. The atoms oscillate around their equilibrium positions, and the interaction between the atoms is responsible for the coupling. The purpose of this study is to highlight and to explain, in a simple way, the dynamic interaction of two coupled oscillators based on normal-modes approach. In addition, at the end of the manuscript the equation of motion is derived based on energy method, and a link is provided to download and play a video in which the phenomenon of coupled oscillator is proven.

Coupled oscillator description

A simple pendulum consists of a mass m hanging from a string of length L and fixed at a pivot point P. When the pendulum is given an initial displacement and released, it will swing back and forth with periodic motion, called *simple harmonic motion*.

Due to its stiffness component the simple pendulum store potential energy being converted to kinetic energy. In the absence of non-conservative forces, this transfer of energy is continual, causing the pendulum to oscillate about its equilibrium. However, non-conservative forces can dissipate or add energy to the system. The friction force is non-conservative and dissipates energy, if the pendulum is given a displacement from equilibrium and released, the energy dissipated by the friction force eventually causes the motion to cease.

Two simple pendulum connected by a horizontal wire, fixed at the same elevation, together they create a *coupled oscillator*. In a coupled oscillator the energy is transferred each other through the wire.

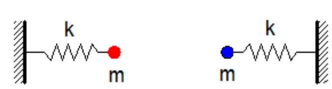


In a coupled oscillator the phenomenon of resonance occurs when the red mass, from the figure above, is given an initial displacement from equilibrium and released from rest, while the blue mass remains at rest. Suddenly, an observer will notice that the blue mass begins to oscillate as time increases, to the point that the pink mass stands still. The blue mass keep on oscillating while the pink mass begins to oscillate once again until the blue mass stops, and so on, and so forth.

From the point of view of work and energy theory [1], the energy flows into the system from one mass to each other through the coupling. The behavior of the coupled oscillator depends entirely on the coupling forces in the spring. If the coupling is strong (high stiffness), then we expect the two masses to oscillate almost in phase. That means, the strong coupling will not allow one mass to lag far behind the motion of the other. If the coupling is weak, then we may expect that the two masses will oscillate, with one sometimes reinforcing the motion of the other (larger amplitudes) and at other times canceling out the motion of the other (smaller amplitudes). So, we can assume the energy is constant due to the energy is being transferred continuously between the two masses.

Normal-mode approach

We can imagine that the wire works like a spring, since both masses are embedded at the same distance, both of them has the same elasticity.



*Este texto, así como el experiemento incluido en el video, fue tomado del material desarrollado durante el curso de vibraciones mecánicas Sem-2017-1.

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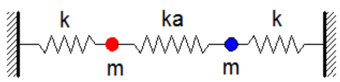
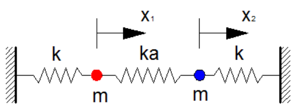
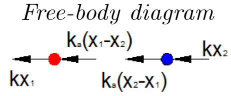
On the other hand, the coupling is given by another wire's section, and it has a different elasticity.



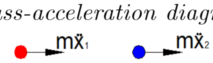
Taken into account the displacements of the masses, we can obtain



Free-body diagram



Mass-acceleration diagram



The spring on the left moved X_1, and the spring on the right moved X_2 while the central one was deformed the difference of these displacements.

From the summation of forces

$$m\ddot{x}_1 = -kx_1 - k_a(x_1 - x_2)$$

$$m\ddot{x}_2 = -kx_2 - k_a(x_2 - x_1) \tag{1}$$

The equation can be expanded as

$$\begin{aligned} m\ddot{x}_1 + kx_1 + k_a x_1 - k_a x_2 = 0 \\ m\ddot{x}_2 + kx_2 + k_a x_2 - k_a x_1 = 0 \end{aligned}$$

Which is rearranged to yield

$$\begin{aligned} m\ddot{x}_1 + (k + k_a)x_1 + k_a x_2 = 0 \\ m\ddot{x}_2 - k_a x_1 + (k + k_a)x_2 = 0 \end{aligned}$$

And writing it in a matrix form leads to

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k + k_a & -k_a \\ -k_a & k + k_a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2}$$

This equation is written in a general form as

$$M\ddot{X} + KX = 0 \tag{3}$$

Eq. (3) is a coupled system of equations. One way to solve this system is to uncouple them by using an appropriate coordinate transformation. One candidate transformation is the modal matrix $[\Phi]$, given by

$$[\Phi] = \left\{ \{X\}_1 \;\; \{X\}_2 \;\; \cdots \;\; \{X\}_N \right\} = \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1N} \\ X_{21} & X_{22} & \cdots & X_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ X_{N1} & X_{N2} & \cdots & X_{NN} \end{bmatrix}$$

where $\{X\}_j$ is the mode shape associated with the jth eigenvalue λ_j^2.

The modal matrix has the desirable property of being orthogonal to the matrices $[M]$ and $[K]$ [2]. Thus, we assume a solution to Eq. (3) of the form

$$\{x(t)\} = \underbrace{[\Phi]}_{\text{Modal matrix}} \; \underbrace{\{\eta(t)\}}_{\text{Modal amplitudes}} \tag{4}$$

where $\{\eta(t)\}$ is a column vector of generalizated (modal) coordinates that are to be determined.

So, we assume a solution of the following form

$$\begin{aligned} x_1 = A_1 \cos \omega t \\ x_2 = A_1 \cos \omega t \end{aligned}$$

Next, we substitute into the differential equation given above

$$\begin{aligned} -mA_1\omega^2 \cos \omega t + (k + k_a)A_1 \cos \omega t - k_a A_2 \cos \omega t = 0 \\ -mA_2\omega^2 \cos \omega t - k_a A_1 \cos \omega t + (k + k_a)A_2 \cos \omega t = 0 \end{aligned}$$

Dividing this equation by $\cos \omega t$

$$\begin{aligned} -A_1\omega^2 m + A_1(k + k_a) - A_2 k_a = 0 \\ -A_2\omega^2 m + A_2(k + k_a) - A_1 k_a = 0 \end{aligned}$$

Which is rearranged to yield

$$\begin{aligned} (k + k_a - \omega^2 m)A_1 - A_2 k_a = 0 \\ -A_1 k_a + (k + k_a - \omega^2 m)A_2 = 0 \end{aligned}$$

In turn, the matrix form is

$$\begin{bmatrix} k + k_a - \omega^2 m & -k_a \\ -k_a & k + k_a - \omega^2 m \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{5}$$

This system of equations has a nontrivial solution for $\{A\}$ only when the determinant of the coefficient matrix is zero. Thus, we arrive at

$$\det\begin{bmatrix} k+k_a-\omega^2 m & -k_a \\ -k_a & k+k_a-\omega^2 m \end{bmatrix} =$$
$$\left(k+k_a-\omega^2 m\right)^2-(-k_a)^2=0$$

Which gives the characteristic equation

$$\omega^4 m^2-(2k+2k_a)m\omega^2+2k_a k+k^2=0$$

Due to the form of this equation, one can treat it as a quadratic polynomial in ω^2, where $\omega^2=\lambda$

$$m^2\lambda^2-(2k+2k_a)m\lambda+2k_a k+k^2=0$$

Solving the equation for $\lambda_{1,2}$

$$\lambda_{1,2}=\frac{k+ka}{m}\pm\frac{ka}{m}$$

Thus, the two roots are given by

$$\omega_1=\sqrt{\lambda_1}=\sqrt{\frac{k+k_a-k_a}{m}}=\sqrt{\frac{k}{m}} \tag{6}$$

$$\omega_2=\sqrt{\lambda_2}=\sqrt{\frac{k+k_a+k_a}{m}}=\sqrt{\frac{k+2k_a}{m}} \tag{7}$$

Which are the two natural frequencies of the system and they have been ordered so that $\omega_1<\omega_2$. That way, the general motion of the coupled oscillator can be considered as the superposition of two normal-modes.

To determine the mode shape associated with ω_1 and ω_2, we set $\omega=\omega_1$ in Eq. (5), to obtain

$$\frac{A_2}{A_1}=\frac{k+k_a-\omega^2 m}{k_a}$$

First mode shape

$$\frac{A_2}{A_1}|_{\omega_1}=\frac{ka}{ka}=1$$

Second mode shape

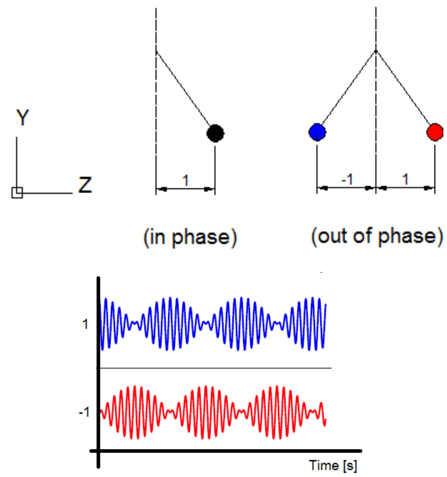
$$\frac{A_2}{A_1}|_{\omega_2}=\frac{-ka}{ka}=-1$$

Therefore, the mode shapes of the coupled oscillator are as follow

$$\{A\}_1=A_1\begin{Bmatrix}1\\1\end{Bmatrix}\quad\text{(in phase)} \tag{8}$$

$$\{A\}_2=A_2\begin{Bmatrix}-1\\1\end{Bmatrix}\quad\text{(out of phase)} \tag{9}$$

Graphically, we can see this behavior as



Energy is initially invested from the blue mass, which is in this instance only weakly coupled to the red mass. As time passes, energy is traded back and forth between the two masess.

Proof of the phenomenon

In order to verify the phenomenon of resonance in a coupled oscillator, please refer to the next link, in which you will be able to play in youtube. (https://youtu.be/WDtkvHRd5UY)

Conclusion

Based on initial conditions and the normal-mode solution of the system, there is a continuous exchange of energy between the two modes, this is due to the phase difference between the two amplitudes. This two amplitudes represents the initial excitation and steady-state response of the system. During one-quarter of the period, the amplitude of an oscillator tends to decay while the other one increases, faster or slower as the system´s resistance allows, resulting in an energy transfer from each other. During the second-quarter of the period the same behavior occurs in the opposite sense, the energy flows in the opposite direction. The process is continually repeated.

Finally, the system´s energy is given by

$$E = E_k + E_p = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 + \frac{1}{2}kx_1^2 + \frac{1}{2}kx_2^2 + \frac{1}{2}k_a(x_2 - x_1)^2$$

Therefore, the total energy of the system is

$$\begin{aligned} E = \quad & \underset{1^{st}\text{ oscillator energy}}{\frac{1}{2}(mv_1^2 + (k + k_a))x_1^2 +} \\ & \underset{2^{nd}\text{ oscillator energy}}{\frac{1}{2}(mv_2^2 + (k + k_a))x_2^2 -} \\ & \underset{\text{Interaction energy}}{k_a x_1 x_2} \end{aligned}$$

References

[1] Haym Benaroya, (2004) Mechanical Vibration: Analysis, Uncertainties and Control, Second Edition, Marcel Dekker.

[2] Balakumar Balachandran, Edward B. Magrab (2009), Vibrations, Second Edition, Cengage Learning.