POSING AND SOLVING LINEAR ALGEBRA AND STATISTICS PROBLEMS WITH MAPLE[1]

Patricia E Balderas-Cañas[2]
Engineering School
The National Autonomous University of Mexico
empatbal@servidor.unam.mx

Abstract

When teaching applied mathematics to engineering graduate students emerge some specific topics that should focus with student's resources or maybe preferences. In this paper, I present the discussion of two problems that were posed and solved by Mexican graduate students in 2005. Those students were enrolled in one of my linear algebra or applied math courses. A main objective of those courses was the students were able to link and use concepts and methods of each subject to formulate and solve engineering problems, and use some CAS to do that. The course was mostly oriented to student acquisition of linear algebra concepts and solving linear programming problems. Those courses were previous to others like optimization, operation research, and decision theory. At the beginning of the course, students where introduced to Maple an Solver (Excel) basics commands to apply linear algebra and statistics tools, then they designed some research-class projects to apply linear algebra concepts and methods. The teaching approach was centered on student interest and the professor role was a promoter of conceptual change (Light, G. y Cox, R., 2001).

The first problem came from a structural analysis in civil engineering, and further discussion with the class was done to approximate a solution of linear systems by the Jacobi's iterative method (see Balderas, 2005) The second problem was posed by the students to link linear algebra with some statistics concepts. Again, further discussion with the whole class was extent to multiple-regression (with a geometric approach of the least squares problem).

Introduction

Mathematics teaching to engineering students challenges both, professors and students, cause communication and goal issues, between others, that we could positively use them to motivate our students' work. Indeed, when we teach applied mathematics to engineering graduate students emerge some specific topics that they should be studied with student's resources and preferences.

In Math Education, let distinguish several sources of posing problems, for example from teacher and professor activities or from senior and graduate students' activities. In this paper my concern is posing and solving problem as student activities. In any place, Balderas and Vazquez

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[2] Full time professor on sabbatical leave in University of Granada, Didactics of Mathematics Department. Sabbatical scholarship provided by Academics Business Office DGAPA of The National Autonomous University of Mexico UNAM, Mexico. September, 2005 – August, 2006.

(2006) presented and discussed some problems that were posed as teaching activities in statistics courses. For the purposes of this paper I analyzed two problems that were posed by Mexican graduate students and solved them as part of research projects developed by the same students.

Below, we first introduce some theoretical elements and present some curricula and teaching details related with of student's posing problems from project developing. Next we describe some problems posed by graduate students and discuss them from students' making decisions approach.

Theoretical elements

Focusing on conceptual understanding in mathematics learning categorized by Hiebert & Lefevre (1986), in particular at the procedural level, students would know how and when use an algorithm, that requires students' acquisition of necessary connections to other concepts of the same or different areas of knowledge. The framework of the study includes elsewhere mathematics education research on intuitions, heuristics and problem solving strategies (NCTM, 1992), as well as students´ knowledge disposition (Prawat, 1989). For example, the study of relative frequencies involved proportional thinking that is a core concept in statistics and calculus courses (Balderas, 2005). Due to that high cognitive demand from students, curriculum designs for engineers must provide links between linear algebra, calculus and statistics courses, as well as other disciplines. And, the elementary courses must set the foundations to have access to higher mathematics concepts.

Indeed, when some teaching goals look for the promotion of student understanding at a generalization and abstraction levels, the teaching expectative is that students would be able to acquire and apply a strong conceptual system.

I think that students' posing problems is a key issue to generalization and abstraction understandings due to many reasons. Firstly, because "… although many statistics students are able to manipulate definitions and algorithms with apparent competence, they often lack understanding of the connections among the important concepts of the discipline they do not know what statistical procedure to apply when they face a real problem of data analysis…" (Schau & Mattern, 1997; cited in Batanero, 2004) Secondly, there is an increasingly easy access to powerful computing facilities that frees time previously devoted to laborious calculations and encourages less formal, more intuitive approaches to statistics (Biehler, 1997; Ben-Zvi, 2000). Thirdly, new course content, new teaching approach and new assessment are required due to this exchange of approaches.

Between other reasons, a recent answer to Moore's question on what skills in the traditional sense should be required (1997) done by Wild & Pfannkuch (1999). But, in spite of Batanero's recommendation (2004) to change the course content and teaching approach at University level, the change is slowly taking place, it seems neither it is produced as soon as work world needs, nor early graduate engineers have a strong statistics background.

For example, there was a four-year discussion to design new curricula for all twelve undergraduate engineering programs in my institution. The final curricula were firstly implemented in the fall semester of 2005. Courses or subjects of each engineering program are grouped into five blocks (basic sciences, engineering sciences, applied engineering, social sciences and other subjects). Statistics and subjects related with it belong to basic sciences block. These subjects are balanced with respect to the total of credits, but not to basic science block (see figure 1). Notice that, only eleven programs are considered in figure 1, they are civil engineering (IC), mining and mechanical engineering (IMM), electronic and electrical engineering (IEE), computation engineering (IC), telecommunication engineering (IT), geological engineering (IGL), geomantic engineering (IGM), industrial engineering (II), mechanical engineering (IM), and petroleum engineering (IP).
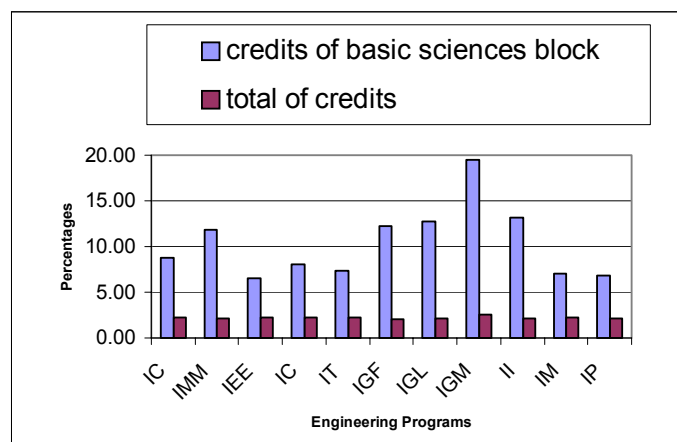

Figure 1. Participation of statistics and related subjects in the total and the basic science block expressed as percentages of credits in engineering curricula of UNAM

Details of teaching context

Six graduate students participant in this study were enrolled in one of my linear algebra (LA), applied math (AM) or statistics (S) classes, each course belongs to graduate mathematics curricula. The undergraduate studies of those participants were pretty close to those showed in the previous section. They were students of civil and systems master engineering programs which require only one math course (6 credits). The rest of curricula (with a total of 72 credits) is grouped into four groups of subjects or activities (knowledge field subjects 18 to 24 credits, complementary subjects 18 to 24 credits, research activities 24 credits, and dissertation work or general examination without credit value).

The teaching approach was centered on students' interest and the professor role was to be a promoter of students' conceptual change, so the teaching goal was developing of students and their conceptions, in such a way that students were responsible of their own learning, the teaching content was built by the students, but they could change their conceptions, and the knowledge was socially constructed (Light, G. y Cox, R., 2001)

The courses were mostly oriented to student acquisition of linear algebra, applied mathematics, and statistics concepts, in CAS settings. The main objective of each course was that the students use concepts and methods of LA, AM and S to formulate and solve engineering problems. In particular, solving linear programming problems was a common topic of linear algebra and applied math courses. The three already mentioned courses were prerequisites of optimization, operation research, and decision theory courses.

At the beginning of each course, students were introduced to Maple9 basics commands of linear algebra, calculus and statistics, and then they designed some research-class projects to apply linear algebra and statistics concepts and methods. Maple9 and Microsoft Excel work sheets were the main CAS used by the students.

Students were asked to develop a project to make a study that uses most of knowledge and resources learnt in linear algebra, applied mathematics or statistics class. The results of each project were presented with the following format: a first page with author and course data, and project title, a second page with the introduction, which must include the formulation of the problem, questions, and objectives of the study. Then, the content and development of the study where knowledge and resources acquired in the course was expressed. It included linear algebra, applied mathematics, and statistical concepts and tools, even those employed for data collection and processing, in order to answer the questions formulated, and according with the objectives of the study.

An important issue in this approach of teaching was the grading of the final report of each project. The assessment of students learning and activities is a huge research area, but for the purposes of this study, I followed some recommendations done by Keeler (1997) and Schau, C. and Mattern, N. (1997). Here, I only point out the aspects and weights (as percentage of the total grade) that were taken into account to grade each report: formulation of problem (15%), questions (15%), and objectives (15%), design of method (20%), development of the study (20%), and elaboration of conclusions (15%).

Discussion of two problems posed and solved by students as class projects

In this study, the analysis of problems posed by students is done with relation to pertinence of students' skills to attend society demands. Accordingly with the engineering curricula in UNAM (2005), engineering students are able to apply scientific and technological knowledge to preserve and improve the environment in a world context. Accordingly with this general characteristic, probability and statistics students, for example, should apply concepts and methods of those subjects to the analysis of random events that occur in the life and the social environment.

The first problem came from a structural analysis in civil engineering. Vicente was a student of my linear algebra course in the fall semester of 2005. The annex 1 is included thanks to his collaboration with this study. He decided to individually develop a short project. His purpose was to point out the importance of linear algebra in structural analysis. In order to do that, he selected two examples where some continuum media principles were additionally applied.

Principles of Continuity, Hooke's law and Equilibrium were used in structural analysis, and linear algebra concepts are such important to express continuity matrixes, rigidities, forces, deformations, displacements, systems of linear equations, and so on. Indeed, eigenvalues and eigenvectors of mass matrix permitted the calculation of vibration modes of a structure. (Annex 1).

Vicente did not pose a research question that could be studied by such linear algebra tools, so the student knowledge of types of problems suggest that he had available several categories of problems as tools to use them in specific situations. Differently, posing a problem from the analysis of a problematic situation when a professional has to solve some engineering problem requires that she/he has a broad repertoire of linear algebra concepts and procedures. It is needed to do more than classify a situation and try to fit some tools to it.

The linear systems generated from the study of civil engineering cases challenged student's knowledge and class discussions, so professor should provide further information or

discuss additional linear algebra topics, as well as, designs programs or didactical material pertinent to the class discussion. For example, I extent the course content to numerical method in order to approximate solutions of specific linear systems. In anywhere, I presented a Maple worksheet elaborated for those reasons. The Jacobi's iterative method was used to approximate solutions of specific linear systems (see Balderas, 2005)

The richness of case studies and project development with graduate students require teaching decisions making to overcome students' difficulties when they develop their projects that really challenge the whole class.

A second problem was thought to minimize labor costs of cashiers at a main highway in Mexico by the optimization of the number of payment cashiers of two-direction highway (Mexico City – Cuernavaca at Tlalpan location, see Annex 2). Judith, Nahum, and Eloy were students of my applied math course in the fall semester of 2005. They were members of a research project team that collaborated with this study.

These students used linear programming and made hypothesis testing of means involved in the model, so they related linear algebra and statistics concepts as a result of their own decision. In this problem was relevant multiple regression, and least squares problem. Students used an estimated attendance rate to simplify the problem and find a solution of the linear programming model. This decision was taken as a result of a whole class discussion. In some sense, students socially negotiated and constructed some pieces of knowledge. However, students did not use binary programming that was either pertinent to the problem[3].

To overcome the team's challenge I discussed some examples of binary and whole programming with the full class, and I extent the discussion of least squares problem with multiple variables to a geometric approach as follows. Those topics were not part of the course content planned at the beginning of the course, and some examples where presented in Maple9® worksheets.

Let $Ax = b$ be a system with m linear equations, where A is a $m \times n$ matrix. We project $m \times 1$ vector b over the column subspace of A. We assume that the number m of observations is greater than the n unknown quantities, that is, we have an inconsistent system. So, there is not likely a selection of x that full fits with b, *i.e.* b is not a linear combination of the columns of A.

So, the problem is to select $\bar{x}$ that minimize the error E, in the sense of least squares. The error $E = \|Ax - b\|$ measures the distance from b to Ax in the column space of A. Remember that

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[3] The students provided the model that appears in the annex 2. A solution was generated with Solver in an Excel® work sheet.

Ax is the linear combination of A columns which coefficients are the coordinates of *x*. Then, finding the solution $\bar{x}$ by least squares, that minimizes the error *E*, is the same that finding the point $p = A\bar{x}$ that is nearer to *b* that ever other point in the column space.

Geometrically, *p* is the projection of *b* over the column space of *A*, and the error vector $Ax - b$ must be orthogonal to that space. The column space of *A* is generated by the column vectors of A, that is A contents all linear combinations of A column vectors.

Let *Ay* be a vector of column space of *A*, the orthogonal condition is expressed by $(Ay)^T(A\bar{x} - b) = 0$ or $y^T[A^T A\bar{x} - A^T b] = 0$. This last must be full fits by every *y*, so the unique possibility is that the vector $A^T A\bar{x} - A^T b = 0$ or $A^T A\bar{x} = A^T b$. Then, the solution in terms of least squares of an inconsistent system of *m* equations with *n* unknowns satisfies $A^T A\bar{x} = A^T b$ (named the normal equations).

If the columns of *A* are linearly independent, the matrix $A^T A$ is invertible and the unique solution of the system by least squares is $\bar{x} = (A^T A)^{-1} A^T b$. In addition, the *b* projection over the column space is $p = A\bar{x} = A(A^T A)^{-1} A^T b$, and the matrix $B = (A^T A)^{-1} A^T$ is a left inverse of *A* due to $BA = (A^T A)^{-1} A^T A = I$.

Conclusions

A main students' reasoning gain in developing projects was that they linked applied mathematics, statistics and linear algebra concepts. It suggests that students' skills are pertinent to solve some social necessities. In that sense, the knowledge acquired by the students is part of the instructional mode of the human inquiry (Keeves and Lakomski 1999).

However, from curricula documents it is not clear whether the core concepts would be discussed with more formalism, and rigor. Some students reach high levels of thinking (see for example Krutetskii, 1976); these levels should be described with detail in teaching documents. From that knowledge, institutional documents (programs and guides) must include learning and teaching recommendations to improve academics activities. And of course, graduate programs must have available relevant bibliography of mathematics education research (learning, teaching, assessment, curricula design, human resources programs, and so on).

Finally, I should mention the lack of methodological foundations of technological and didactical approaches commonly used in graduate courses; most of them emerge from teaching

experiences instead of teaching experiments. Thanks to this type of symposia, where the exchange of teaching approaches is the focus, some mathematics professors that are not mathematics education researches, could systematize their own teaching experience.

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Annex 1

Next, I synthesized and translated the examples reported by Vicente as the result of the development of his project. He reported the bibliography that appears at the end of two examples.

Example 1

From the analysis of the forces and displacements in the two free knots of a structure that are pointed in the figure 1, the linear equation (1) for deformation vector is built, and its solution is necessary to find mass deformations due to the principle of continuity.
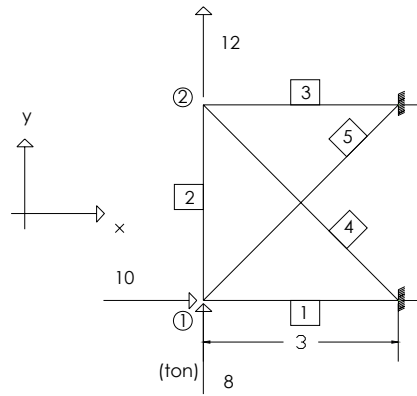

Figure 1

We separately study each bar when positive displacements in some direction (which value falls between 0° and 90°) are applied in each knot. The applied force produces the displacement (d_1), so the projections over Cartesian axes are

$$d_{x_1} = d_1 \cos(\theta)$$
$$d_{y_1} = d_1 \sin(\theta)$$

The angle for bar 1 is $\theta = 0°$, so the projection of both components of last displacements aloud us to express the deformation e of the bar 1 as

$$e_1 = \delta = -d_{x_1}$$

Similarly, the rest of deformations are gotten. For bar 2, the displacements d_1 and d_2 are applied, where $\theta = 0°$, so $\delta = d_{y_2} - d_{y_1}$, $\Delta = d_{x_2} - d_{x_1}$, and thus $e_2 = d_{y_2} - d_{y_1}$

The bar 3 is similar to bar 1, but it is function of de knot two, so $e_3 = \delta = -d_{x_2}$

Accordingly with figure 2, the deformation for the bar 4 is
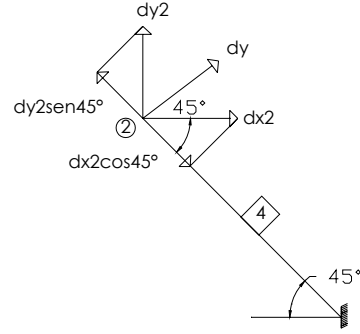
$$e_4 = -d_{x_2} \cos 45° + d_{y_2} \sin 45°$$

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Figure 2

We similarly proceed for bar 5, in terms of knot 1 displacement

$$e_5 = -d_{x_1} \cos 45° - d_{y_1} \sin 45°$$

Then, due to continuity principle, the relation between displacements and deformations is expressed by the matrix equation

$$\{e\} = [A]\{d\} \text{ or}$$

$$\begin{Bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{Bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -0.71 & 0.71 \\ -0.71 & -0.71 & 0 & 0 \end{bmatrix} \begin{Bmatrix} d_{x_1} \\ d_{y_1} \\ d_{x_2} \\ d_{y_2} \end{Bmatrix}$$

In this example, we assume that the rigidities of the bars are as follows

$$k_1 = k_4 = 2 \ ton/cm$$

$$k_2 = k_3 = k_5 = 3 \ ton/cm$$

Thus, the correspondent matrix of rigidities of each bar is

$$k = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

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Then, we build the matrix of rigidity with the matrix of continuity and the matrix of bar rigidities:

$$[A]^T [k][A] = [K]$$

So, after calculations we get that matrix of rigidity:

$$[K] = \begin{bmatrix} 3.5 & 1.5 & 0 & 0 \\ 1.5 & 4.5 & 0 & -3 \\ 0 & 0 & 4 & -1 \\ 0 & -3 & -1 & 4 \end{bmatrix}$$

The vector of forces (in tons), $\{F\}$, that it is associated with the structure, in this example, has the following components

$$\{F\} = \begin{Bmatrix} 10 \\ 8 \\ 0 \\ 12 \end{Bmatrix}$$

Then, the displacement of knots are gotten from the solution of the equation $\{F\} = [K]\{d\}$ that takes the following shape in this example

$$\begin{Bmatrix} 10 \\ 8 \\ 0 \\ 12 \end{Bmatrix} = \begin{bmatrix} 3.5 & 1.5 & 0 & 0 \\ 1.5 & 4.5 & 0 & -3 \\ 0 & 0 & 4 & -1 \\ 0 & -3 & -1 & 4 \end{bmatrix} \{d\} \tag{1}$$

Solving the equation by gauss-jordan method[4] the displacement vector is the following

$$\{d\} = \begin{Bmatrix} -1.059 \\ 9.137 \\ 2.627 \\ 10.51 \end{Bmatrix}$$

By the substitution of this displacement vector in the equation $\{e\} = [A]\{d\}$ produces the element values (in cm) of the deformation vector:

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[4] The solution was calculated with Maple9®.

$$\{e\} = \begin{Bmatrix} 1.059 \\ 1.373 \\ -2.627 \\ 5.574 \\ -5.712 \end{Bmatrix}$$

Now, we apply the Hooke's law to calculate the vector of internal forces $\{P\} = [k]\{e\}$.

$$\{P\} = \begin{Bmatrix} 2.118 \\ 4.119 \\ -7.881 \\ 11.148 \\ -17.136 \end{Bmatrix}$$

In the figure 3 are shown the forces that are acting over the structure. Their positive values mean tension, and the negative ones compression over the respective knot.
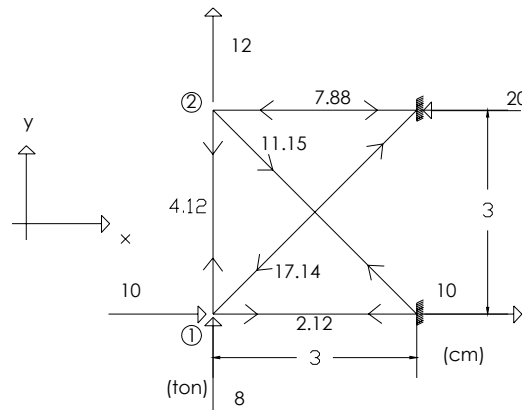

Figure 3

The reactions are directly gotten from the concurrent forces in the support points.

Finally, we test the equilibrium of the structure with the expressions

$$\sum F_x = 0$$
$$\sum F_y = 0$$

For knot 1: $\qquad \sum F_y = 8 + 4.12 + 17.14 \sin 45° = 0$

For knot 2:
$$\sum F_x = -7.88 + 11.15 \cos 45° = 0 \quad \text{and}$$
$$\sum F_y = -4.12 + 12 - 11.15 \sin 45° = 0$$

Example 2

This example illustrates the calculation of vibration modes in a structure based on the calculation of proper values or eigenvalues. The figure 4 shows schemata of a building. This building has five plants. The structure is symmetric and totally regular, mass and torsion centers are coincident. Thus, the building is analyzed by two independent and orthogonal models. Each model has one degree of freedom per plant. The frame is made with strength concrete.

Figure 4. Dynamic model of a five-plants building
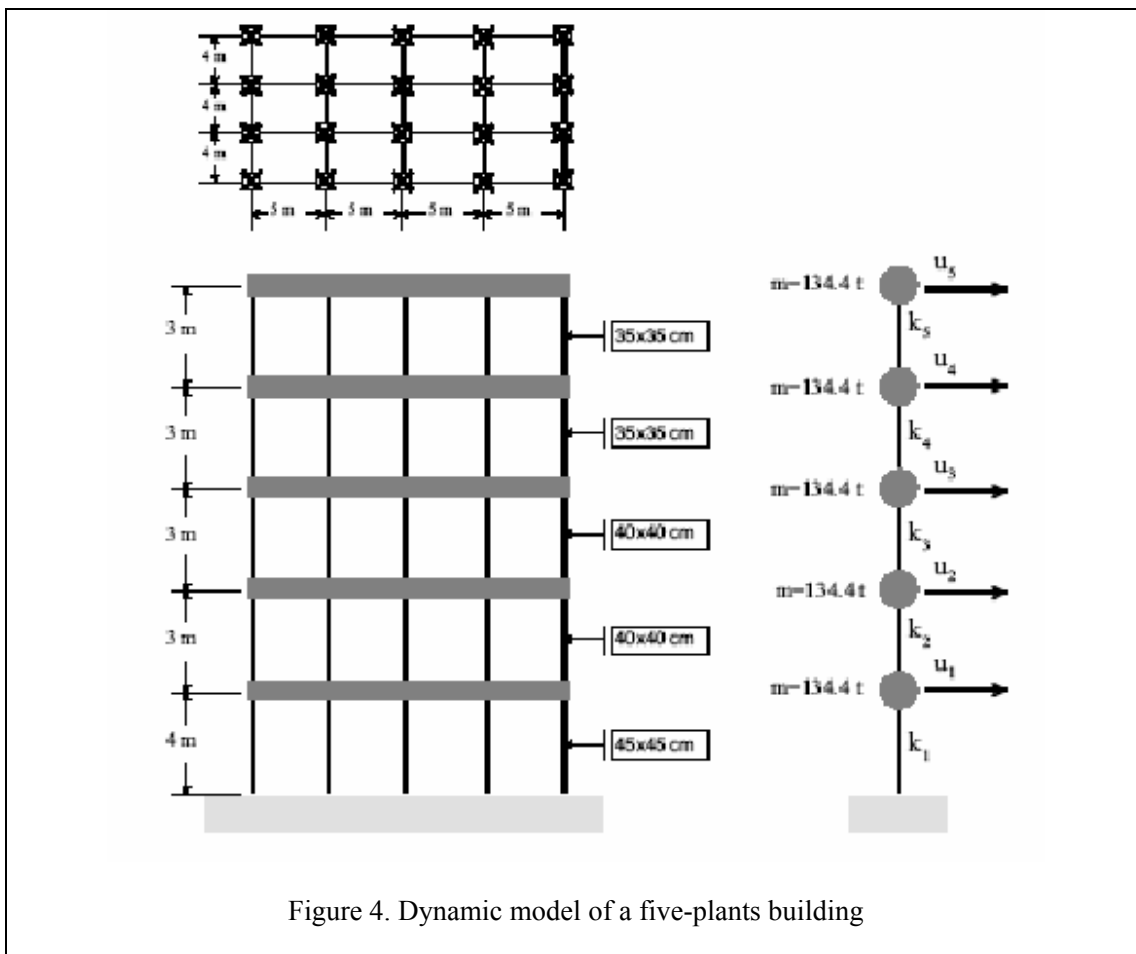
Mass calculation

Only one type of plant is considered, even for the cover.

Permanent charge: $G = 500 \; Kg/m^2$

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Overcharge used: $Q = 200\ Kg/m^2$

Mass calculated:

$$m_k = (20\ m \times 12\ m) \times (500\ Kg/m^2 + 0.3 \times 200\ Kg/m^2) = 134{,}400\ Kg$$

Proper periods and vibration modes

The rigidity due to transversal displacement of pilots is obtained for each plant as follow:

$$k_1 = 20 \times k_{c_1} = 20 \times \frac{12EI_{c_1}}{L_{c_1}} = 2.56 \times 10^8\ N/m$$

$$k_2 = k_3 = 20 \times k_{c_2} = 20 \times \frac{12EI_{c_2}}{L_{c_2}} = 3.79 \times 10^8\ N/m$$

$$k_4 = k_5 = 20 \times k_{c_4} = 20 \times \frac{12EI_{c_4}}{L_{c_4}} = 2.22 \times 10^8\ N/m$$

Then, the matrix of rigidity is

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 & 0 & 0 \\ -k_2 & k_2 + k_3 & -k_3 & 0 & 0 \\ 0 & -k_3 & k_3 + k_4 & -k_4 & 0 \\ 0 & 0 & -k_4 & k_4 + k_5 & -k_3 \\ 0 & 0 & 0 & -k_3 & k_5 \end{bmatrix} = \begin{bmatrix} 6.35 & -3.79 & 0 & 0 & 0 \\ -3.79 & 7.58 & -3.79 & 0 & 0 \\ 0 & -3.79 & 6.01 & -2.22 & 0 \\ 0 & 0 & -2.22 & 4.44 & -2.22 \\ 0 & 0 & 0 & -2.22 & 2.22 \end{bmatrix} \times 10^8$$

The mass matrix is diagonal and it is expressed as

$$M = \begin{bmatrix} 134400 & 0 & 0 & 0 & 0 \\ 0 & 134400 & 0 & 0 & 0 \\ 0 & 0 & 134400 & 0 & 0 \\ 0 & 0 & 0 & 134400 & 0 \\ 0 & 0 & 0 & 0 & 134400 \end{bmatrix}$$

The natural frequencies are gotten from the characteristic equation

$$\left| K - \omega^2 M \right| = 0$$

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Which eigenvalues are $\omega_1^2 = 180.1$, $\omega_2^2 = 125.38$, $\omega_3^2 = 3601.4$, $\omega_4^2 = 5518.2$, and $\omega_5^2 = 9257.4$.

Thus, the proper periods of the structure are $T_1 = 0.468\ s$, $T_2 = 0.177\ s$, $T_3 = 0.105\ s$, $T_4 = 0.085\ s$, and $T_5 = 0.065\ s$.

Finally, the vibration modes are the eigenvectors of the system:

$$\phi_1 = \begin{Bmatrix} 0.199 \\ 0.321 \\ 0.422 \\ 0.549 \\ 0.616 \end{Bmatrix} \quad \phi_2 = \begin{Bmatrix} 0.440 \\ 0.542 \\ 0.403 \\ -0.139 \\ -0.576 \end{Bmatrix} \quad \phi_3 = \begin{Bmatrix} -0.528 \\ -0.211 \\ 0.375 \\ 0.558 \\ -0.474 \end{Bmatrix} \quad \phi_4 = \begin{Bmatrix} 0.529 \\ -0.148 \\ -0.535 \\ 0.590 \\ -0.252 \end{Bmatrix} \quad \phi_5 = \begin{Bmatrix} 0.457 \\ -0.733 \\ 0.482 \\ -0.143 \\ 0.031 \end{Bmatrix}$$

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Annex 2

Next, I synthesized and translated the report of Judith, Nahum, and Eloy as the result the development of their project. This team reported the bibliography that appears after the linear programming model. They would optimize the number of payment cashiers of two-direction highway Mexico City – Cuernavaca at Tlalpan location. The number of cars that passed through the payment cashiers in every time period and direction, were tested on the basis of rough data provided by (CAPUFE, 2005). For each time period (table 1) and direction (A: Mexico City to Cuernavaca; B: Cuernavaca to Mexico City), a 45 days sample of the number of cars was taken per time period and direction (tables 2 and 3).

Table 1. Time period code

Period	Code	From:	To:
1	H1	01:00	05:00
2	H2	05:00	09:00
3	H3	09:00	13:00
4	H4	13:00	17:00
5	H5	17:00	21:00
6	H6	21:00	01:00

Table 2. Mean μ, sample mean ($\bar{x}$) and standard deviation,
and standardized normal values (Mexico City to Cuernavaca)

A		μ	$\bar{x}$	s	z_x
Period	1	318	322	30.0	0.894
	2	1,272	1,267	130.7	-0.257
	3	2,592	2,639	316.4	0.996
	4	3,011	3,073	302.6	1.375
	5	4,273	4,240	501.6	-0.441
	6	2,045	2,012	155.9	-1.420

Table 3. Mean μ, sample mean ($\bar{x}$) and standard deviation,
and standardized normal values (Cuernavaca to Mexico City)

B		μ	$\bar{x}$	s	z_x
Period	1	339	334	35.16	-0.954
	2	1,077	1,060	116.21	-0.981
	3	1,912	1,928	199.74	0.537
	4	2,150	2,125	190.55	-0.880
	5	2,314	2,351	232.39	1.068
	6	1,017	1,019	104.16	0.129

A test hypothesis of each number of cars was made. The next two tables content the normal standardized values and the decision of null hypothesis acceptance with a significance level of 0.1. Table 4 with data for the direction A, and table 5 with data for the opposite direction B.

Table 4. Standardized values and decisions of acceptance
the null hypothesis for the direction A

A		$-z_{\alpha/2}$	z_x	$z_{\alpha/2}$	Acceptance
Period	1	1.6449	**0.89369**	1.6449	Yes
	2	1.6449	**-0.25659**	1.6449	Yes
	3	1.6449	**0.99648**	1.6449	Yes
	4	1.6449	**1.3746**	1.6449	Yes
	5	1.6449	**-0.44137**	1.6449	Yes
	6	1.6449	**-1.4202**	1.6449	Yes

Table 5. Standardized values and decisions of acceptance
the null hypothesis for the direction B

B		$-z_{\alpha/2}$	z_x	$z_{\alpha/2}$	Acceptance
Period	1	-1.6449	-0.95389	1.6449	Yes
	2	-1.6449	-0.9813	1.6449	Yes
	3	-1.6449	0.53734	1.6449	Yes
	4	-1.6449	-0.88009	1.6449	Yes
	5	-1.6449	1.068	1.6449	Yes
	6	-1.6449	0.12881	1.6449	Yes

With those means of cars that pass through each payment cashier per time period, a linear programming model is designed to optimize de number of open payment cashiers in each direction that full fit the requirements of attendance that produce a minimum cost of labor.

Linear Programming Model[5]

Total car attendance per time period in each payment cashier is 1400 cars

Variable definition

Labor costs:
CC_{H1} y CC_{H6} = Labor costs for periods 1 and 6 = 300.00 pesos
CC_{H2}, CC_{H3}, CC_{H4} y CC_{H5} = Labor costs for periods 2, 3, 4 and 5 = 200.00 pesos

The number of cashiers per time period:

NC_{H1} = Cashiers direction A + Cashiers direction B time period 1
NC_{H2} = Cashiers direction A + Cashiers direction B time period 2
NC_{H3} = Cashiers direction A + Cashiers direction B time period 3
NC_{H4} = Cashiers direction A + Cashiers direction B time period 4
NC_{H5} = Cashiers direction A + Cashiers direction B time period 5
NC_{H6} = Cashiers direction A + Cashiers direction B time period 6

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[5] The students provided a solution generated with Solver in an Excel® work sheet different to the solution that appears at the end of this annex. The solution here included corresponds to the linear programming model that they built. The students' solution was modified by additional information of attendance rate of each cashier that they did not consider in the model.

CA_{H1} = Open cashiers direction A time period 1
CA_{H2} = Open cashiers direction A time period 2
CA_{H3} = Open cashiers direction A time period 3
CA_{H4} = Open cashiers direction A time period 4
CA_{H5} = Open cashiers direction A time period 5
CA_{H6} = Open cashiers direction A time period 6

CB_{H1} = Open cashiers direction B time period 1
CB_{H2} = Open cashiers direction B time period 2
CB_{H3} = Open cashiers direction B time period 3
CB_{H4} = Open cashiers direction B time period 4
CB_{H5} = Open cashiers direction B time period 5
CB_{H6} = Open cashiers direction B time period 6

Objective function:

Min Z = 300 ($NC_{H1} + NC_{H6}$) + 200 ($NC_{H2} + NC_{H3} + NC_{H4} + NC_{H5}$)

Constraints: (estimated by the team)

920 $CA_{H1} \geq 318$
920 $CA_{H2} \geq 1272$
920 $CA_{H3} \geq 2592$
920 $CA_{H4} \geq 3011$
920 $CA_{H5} \geq 4273$
920 $CA_{H6} \geq 2045$
920 $CB_{H1} \geq 339$
920 $CB_{H2} \geq 1077$
920 $CB_{H3} \geq 1912$
920 $CB_{H4} \geq 2150$
920 $CB_{H5} \geq 2314$
920 $CB_{H6} \geq 1017$

$NC_{H1,} NC_{H2,} NC_{H3,} NC_{H4,} NC_{H5,} NC_{H6} \leq 13$
$CA_{H1,} CA_{H2,} CA_{H3,} CA_{H4,} CA_{H5,} CA_{H6} \leq 8$
$CB_{H1,} CB_{H2,} CB_{H3,} CB_{H4,} CB_{H5,} CB_{H6} \leq 11$
$CA_{H1,} CA_{H2,} CA_{H3,} CA_{H4,} CA_{H5,} CA_{H6,} CB_{H1,} CB_{H2,} CB_{H3,} CB_{H4,} CB_{H5,} CB_{H6} \geq 1$

Bibliography reported by the team

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Time period (Hrs.)	Attendance A		Minimum attendance A	Attendance B		Minimun attendance B
01 – 05	920	>=	318	920	>=	339
05 – 09	1840	>=	1272	1840	>=	1077
09 – 13	2760	>=	2592	2760	>=	1912
13 – 17	3680	>=	3011	2760	>=	2150
17 – 21	4600	>=	4273	2760	>=	2314
21 – 01	2760	>=	2045	1840	>=	1017

Time period	Cost per period	Cost A ($)	Cost B ($)
01 – 05	300	300	300
05 – 09	200	400	400
09 – 13	200	600	600
13 – 17	200	800	600
17 – 21	200	1000	600
21 – 01	300	900	600

Total cost ($)

$7.100,00

Time period	# Cashiers A	# Cashiers B	Total of cashiers		Maximun of cashiers
01 – 05	1,00	1,00	2	<=	13
05 – 09	2,00	2,00	4	<=	13
09 – 13	3,00	3,00	6	<=	13
13 – 17	4,00	3,00	7	<=	13
17 – 21	5,00	3,00	8	<=	13
21 – 01	3,00	2,00	5	<=	13

Time period	Variable # cashiers A		Total of cashiers A	Variable # cashiers B		Total of csashiers B
01 – 05	1,00	<=	8	1,00	<=	11
05 – 09	2,00	<=	8	2,00	<=	11
09 – 13	3,00	<=	8	3,00	<=	11
13 – 17	4,00	<=	8	3,00	<=	11
17 – 21	5,00	<=	8	3,00	<=	11
21 – 01	3,00	<=	8	2,00	<=	11

Time period	# cashiers A		Minimum of cashiers A	# cashiers B		Minimum of cashiers B
01 – 05	1,00	>=	1	1,00	>=	1
05 – 09	2,00	>=	1	2,00	>=	1
09 – 13	3,00	>=	1	3,00	>=	1
13 – 17	4,00	>=	1	3,00	>=	1
17 – 21	5,00	>=	1	3,00	>=	1
21 – 01	3,00	>=	1	2,00	>=	1