El potencial electrostático es: $\Phi = \frac{1}{4\pi\varepsilon_0}\int_{V'} dV' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$

Para el potencial en puntos lejanos, debemos determinar el desarrollo en serie de Taylor de: $\frac{1}{|\vec{r}-\vec{r}'|}$ en la vecindad de $\vec{r}' = 0$.

Entonces, sea $\varphi(\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|}$

$\varphi(\vec{r}_* + \Delta\vec{r}_*) = \varphi(\vec{r}')$, con $\vec{r}_* \to 0$ y $\Delta\vec{r}_* = \vec{r}'$

Pero de acuerdo con la ecuación (*) - ver archivo anterior:

$\varphi(\vec{r}_* + \Delta\vec{r}_*) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\Delta\vec{r}_*\bullet\nabla^*\right)^n \varphi(\vec{r}_*)$ donde $\varphi(\vec{r}_*) = \frac{1}{|\vec{r}-\vec{r}_*|}$

$$\Rightarrow \varphi(\vec{r}') = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\vec{r}'\bullet\nabla^*\right)^n \varphi(\vec{r}_*)\Big|_{\vec{r}_*=0}$$

Desarrollando los primeros tres términos de la suma:

- **Para n = 0:**

$$\frac{1}{0!}\left(\vec{r}'\bullet\nabla^*\right)^0 \varphi(\vec{r}_*)\Big|_{\vec{r}_*=0} = \varphi(\vec{r}_*)\Big|_{\vec{r}_*=0} = \frac{1}{|\vec{r}-0|} = \frac{1}{r}$$

- **Para n = 1:**

$$\frac{1}{1!}\left(\vec{r}'\bullet\nabla^*\right)^1 \varphi(\vec{r}_*)\Big|_{\vec{r}_*=0} = \left(\vec{r}'\bullet\nabla^*\right)\varphi(\vec{r}_*)\Big|_{\vec{r}_*=0} = \vec{r}'\bullet\left[\nabla^*\frac{1}{|\vec{r}-\vec{r}_*|}\right]_{\vec{r}_*=0}$$

$$= \vec{r}'\bullet\left[\frac{\vec{r}-\vec{r}_*}{|\vec{r}-\vec{r}_*|^3}\right]_{\vec{r}_*=0} = \vec{r}'\bullet\left[\frac{\vec{r}}{r^3}\right] = \frac{\vec{r}'\bullet\vec{r}}{r^3}$$

- **Para n = 2:**

$$\frac{1}{2!}\left(\vec{r}'\bullet\nabla^*\right)^2 \varphi(\vec{r}_*)\Big|_{\vec{r}_*=0} = \frac{1}{2!}\vec{r}'\bullet\nabla^*\left[\vec{r}'\bullet\left(\frac{\vec{r}-\vec{r}_*}{|\vec{r}-\vec{r}_*|^3}\right)\right]_{\vec{r}*=0} = \frac{1}{2!}\vec{r}'\bullet\nabla^*\left[\frac{\vec{r}'\bullet\vec{r}-\vec{r}'\bullet\vec{r}_*}{|\vec{r}-\vec{r}_*|^3}\right]_{\vec{r}_*=0}$$

$$= \frac{1}{2!}\vec{r}'\bullet\left[\frac{1}{|\vec{r}-\vec{r}_*|^3}\left(\nabla^*\left(\vec{r}'\bullet\vec{r}\right) - \nabla^*\left(\vec{r}'\bullet\vec{r}_*\right)\right) + \left(\vec{r}'\bullet\vec{r}-\vec{r}'\bullet\vec{r}_*\right)\nabla^*\frac{1}{|\vec{r}-\vec{r}_*|^3}\right]_{\vec{r}_*=0}$$

$$=\frac{1}{2!}\vec{r}'\bullet\left[\frac{-\vec{r}'}{|\vec{r}-\vec{r}_*|^3}+3\frac{(\vec{r}'\bullet\vec{r}-\vec{r}'\bullet\vec{r}_*)(\vec{r}-\vec{r}_*)}{|\vec{r}-\vec{r}_*|^5}\right]_{\vec{r}_*=0}=\frac{1}{2!}\vec{r}'\bullet\left[-\frac{\vec{r}'}{r^3}+\frac{3(\vec{r}'\bullet\vec{r})\vec{r}}{r^5}\right]$$

$$=\frac{1}{2}\vec{r}'\bullet\left[\frac{3(\vec{r}'\bullet\vec{r})\vec{r}}{r^5}-\frac{\vec{r}'}{r^3}\right]=\frac{1}{2}\left[\frac{3(\vec{r}'\bullet\vec{r})^2}{r^5}-\frac{r'^2}{r^3}\right]=\frac{1}{2}\left[\frac{3(\vec{r}'\bullet\vec{r})^2-r'^2r^2}{r^5}\right]$$

Entonces:

$$\frac{1}{|\vec{r}-\vec{r}'|}\approx\sum_{n=0}^{2}\frac{1}{n!}\left(\vec{r}'\bullet\nabla^*\right)^n\varphi(\vec{r}_*)\Big|_{\vec{r}_*=0}=\frac{1}{r}+\frac{\vec{r}'\bullet\vec{r}}{r^3}+\frac{1}{2}\left[\frac{3(\vec{r}'\bullet\vec{r})^2-r'^2r^2}{r^5}\right]$$

Sustituyendo en el potencial:

$$\Phi\approx\frac{1}{4\pi\varepsilon_0}\int_{V'}dV'\frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}=\frac{1}{4\pi\varepsilon_0}\int_{V'}dV'\rho(\vec{r}')\left(\frac{1}{r}+\frac{\vec{r}'\bullet\vec{r}}{r^3}+\frac{1}{2}\left[\frac{3(\vec{r}'\bullet\vec{r})^2-r'^2r^2}{r^5}\right]\right)$$

$$\approx\frac{1}{4\pi\varepsilon_0 r}\int_{V'}dV'\rho(\vec{r}')+\frac{1}{4\pi\varepsilon_0 r^3}\vec{r}\bullet\int_{V'}dV'\vec{r}'\rho(\vec{r}')+\frac{1}{8\pi\varepsilon_0 r^5}\int_{V'}dV'\rho(\vec{r}')\left[3(\vec{r}'\bullet\vec{r})^2-r'^2r^2\right]$$

Pero en la tercera integral:

$$\left[3(\vec{r}'\bullet\vec{r})^2-r'^2r^2\right]=3\sum_i x_ix_i'\sum_j x_jx_j'-r'^2\sum_i x_ix_i=3\sum_{i,j}x_ix_i'x_jx_j'-r'^2\sum_{i,j}\delta_{ij}x_ix_j$$

$$\Rightarrow\int_{V'}dV'\rho(\vec{r}')\left[3(\vec{r}'\bullet\vec{r})^2-r'^2r^2\right]=\int_{V'}dV'\rho(\vec{r}')\left[\sum_{i,j}3x_ix_i'x_jx_j'-r'^2\sum_{i,j}\delta_{ij}x_ix_j\right]$$

$$=\sum_{i,j}x_ix_j\int_{V'}dV'\rho(\vec{r}')\left[3x_i'x_j'-r'^2\delta_{ij}\right]$$

Por lo tanto:

$$\Phi\approx\frac{1}{4\pi\varepsilon_0 r}\underbrace{\int_{V'}dV'\rho(\vec{r}')}_{\substack{\text{carga } total\ (monopolo)\\ Q_T}}+\frac{1}{4\pi\varepsilon_0 r^3}\vec{r}\bullet\underbrace{\int_{V'}dV'\vec{r}'\rho(\vec{r}')}_{\substack{momento\ dipolar\\ \vec{p}}}+\frac{1}{8\pi\varepsilon_0 r^5}\sum_{i,j}x_ix_j\underbrace{\int_{V'}dV'\rho(\vec{r}')\left[3x_i'x_j'-r'^2\delta_{ij}\right]}_{\substack{momento\ cuadrupolar\\ Q_{ij}}}$$

Las integrales representan los momentos de la distribución de carga

Transformando

$$\int_{V'}dV'\leftrightarrow\sum_n$$

$$\rho(\vec{r}')\leftrightarrow q_n$$

Distribución	continua	discreta	
1. Carga total o monopolo Q_T	$\int_{V'} dV' \rho(\vec{r}')$	$\sum_n q_n$	escalar
2. Momento dipolar $\vec{p}$	$\int_{V'} dV' \vec{r}' \rho(\vec{r}')$	$\sum_n \vec{r}_n' q_n$	vector
3. Momento cuadrupolar Q_{ij}	$\int_{V'} dV' \rho(\vec{r}')\left[3x_i' x_j' - r'^2 \delta_{ij}\right]$	$\sum_n q_n \left[3x_i' x_j' - r'^2 \delta_{ij}\right]$	tensor

$$\Rightarrow \Phi \approx \frac{1}{4\pi\varepsilon_0 r} Q_T + \frac{1}{4\pi\varepsilon_0 r^3} \vec{r} \bullet \vec{p} + \frac{1}{8\pi\varepsilon_0 r^5} \sum_{i,j} x_i Q_{ij} x_j$$

$$\Rightarrow \Phi \approx \frac{1}{4\pi\varepsilon_0} \frac{Q_T}{r} + \frac{1}{4\pi\varepsilon_0} \frac{\vec{p} \bullet \vec{r}}{r^3} + \frac{1}{8\pi\varepsilon_0} \frac{\vec{r}^T \hat{Q} \vec{r}}{r^5}$$

donde:

$$\vec{r}^T \hat{Q} \vec{r} = \sum_{i,j} x_i Q_{ij} x_j = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

con

$$Q_{ij} = \int_{V'} dV' \rho(\vec{r}')\left[3x_i' x_j' - r'^2 \delta_{ij}\right] \quad o \quad \sum_n q_n \left[3x_i' x_j' - r'^2 \delta_{ij}\right]$$

De la definición del tensor cuadrupolar, podemos notar las siguientes propiedades del tensor:

- El tensor es simétrico: $Q_{ij} = Q_{ji}$
- La traza (suma de los elementos diagonales) es igual a cero:

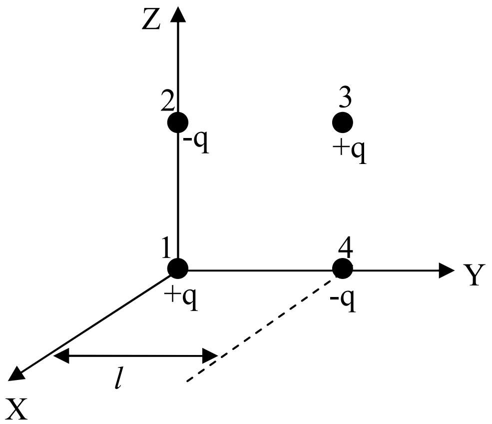
$$\sum_i Q_{ii} = \int_{V'} dV' \rho(\vec{r}') \sum_i \left[3x_i' x_i' - r'^2 \delta_{ii}\right] = \int_{V'} dV' \rho(\vec{r}') \sum_i \left[3x_i'^2 - r'^2\right]$$
$$= \int_{V'} dV' \rho(\vec{r}')\left[3\left(x_1'^2 + x_2'^2 + x_3'^2\right) - 3r'^2\right] = 0$$

$$\Rightarrow Q_{11} + Q_{22} + Q_{33} = 0$$

$\Rightarrow \hat{Q}$ solo tiene 5 elementos independientes, tres de los elementos no diagonales y dos por la condición de la traza.

Ejemplo:

Calcular el potencial que crean a grandes distancias cuatro cargas puntuales situadas en el plano YZ de forma que ocupan los vértices de un cuadrado de lado *l*. Las cargas tienen magnitudes iguales y signos alternados:



momento monopolar:

$$Q_T = \sum_n q_n = +q - q + q - q = 0$$

momento dipolar:

$$\vec{p} = \sum_n \vec{r}_n' q_n = 0(q) + l\hat{z}(-q) + l(\hat{y} + \hat{z})(q) + l\hat{y}(-q) = 0$$

n	q_n	x_1'	x_2'	x_3'	r'
1	+q	0	0	0	0
2	- q	0	0	l	l
3	+q	0	l	l	$\sqrt{2}\, l$
4	- q	0	l	0	l

momento cuadrupolar:

$$Q_{ij} = \sum_n q_n \left[3x_i' x_j' - r'^2 \delta_{ij} \right]$$

$$Q_{11} = \sum_n q_n \left[3x_1' x_1' - r'^2 \right] = q(0) - q(-l^2) + q(-2l^2) - q(-l^2) = ql^2 - 2ql^2 + ql^2 = 0$$

$$Q_{22} = \sum_n q_n \left[3x_2' x_2' - r'^2 \right] = q(0) - q(-l^2) + q(3l^2 - 2l^2) - q(3l^2 - l^2) = ql^2 + ql^2 - 2ql^2 = 0$$

Como la traza es igual a cero: $Q_{11} + Q_{22} + Q_{33} = 0$

y en este ejemplo: $Q_{11} = Q_{22} = 0$

Entonces

$Q_{33} = 0$

Ahora los elementos fuera de la diagonal:

$$Q_{12} = Q_{21} = \sum_n q_n \left[3x_1' x_2' \right] = 0$$

$$Q_{13} = Q_{31} = \sum_n q_n \left[3x_1' x_3' \right] = 0$$

$$Q_{23} = Q_{32} = \sum_n q_n \left[3x_2' x_3' \right] = q(3l^2) = 3ql^2$$

Entonces: $\hat{Q} = \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 3ql^2 \\ 0 & 3ql^2 & 0 \end{pmatrix}$

Y

$$\vec{r}^{\,T}\hat{Q}\vec{r} = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}\begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 3ql^2 \\ 0 & 3ql^2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 3ql^2 z & 3ql^2 y \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3ql^2 yz + 3ql^2 yz = 6ql^2 yz$$

$$\Rightarrow \Phi \approx \frac{1}{4\pi\varepsilon_0}\frac{Q_T}{r} + \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\bullet\vec{r}}{r^3} + \frac{1}{8\pi\varepsilon_0}\frac{\vec{r}^{\,T}\hat{Q}\vec{r}}{r^5} = 0 + 0 + \frac{1}{8\pi\varepsilon_0}\frac{6ql^2 yz}{r^5}$$

$$\Rightarrow \Phi \approx \frac{1}{8\pi\varepsilon_0}\frac{6ql^2 yz}{r^5} = \frac{1}{4\pi\varepsilon_0}\frac{3ql^2 yz}{r^5}$$

$$usando\ coordenadas\ esféricas: \begin{cases} y = r\,sen\theta\, sen\varphi \\ z = r\cos\theta \end{cases}$$

$$\Rightarrow \Phi \approx \frac{1}{4\pi\varepsilon_0}\frac{3ql^2 r^2 sen\theta\cos\theta sen\varphi}{r^5} = \frac{1}{4\pi\varepsilon_0}\frac{3ql^2 sen\theta\cos\theta sen\varphi}{r^3} = \frac{1}{4\pi\varepsilon_0}\frac{3ql^2 sen2\theta sen\varphi}{2r^3}$$

la última igualdad es debido a que: $sen2\theta = 2sen\theta\cos\theta$

$$\therefore \Phi \approx \frac{1}{4\pi\varepsilon_0}\frac{3ql^2}{2r^3} sen2\theta sen\varphi$$