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Optimal constants of L^2 inequalities for closed nearly umbilical hypersurfaces in space forms

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Abstract Let Σ be a smooth closed hypersurface with non-negative Ricci curvature, isometrically immersed in a space form. It has been proved in Cheng (Pacific J Math, 2014), Cheng and Zhou (J Geom Anal, 2012), Perez (On nearly umbilical hypersurfaces, 2011) that there are some L^2 inequalities on Σ which measure the stability of closed umbilical hypersurfaces or more generally, closed hypersurfaces with traceless Newton transformation of the second fundamental form. In this paper, we prove that the constants in these inequalities are optimal.

Keywords L^2 stability inequality · Closed umbilical hypersurfaces · Newton transformation · The second fundamental form

Mathematics Subject Classification 53C42 · 49Q10

1 Introduction

A hypersurface Σ is called totally umbilical if its second fundamental form A is multiple of its metric g at every point, that is, $A = \frac{\mathrm{tr}A}{n} g$ on Σ. In differential geometry, a classical theorem states that a closed, i.e., compact and without boundary, totally umbilical surface isometrically immersed in the Euclidean space $\mathbb{R}^3$ must be a round sphere $\mathbb{S}^2$ and in particular, its second fundamental form A is a constant multiple of the metric. This result is also true for hypersurfaces in $\mathbb{R}^{n+1}$.

It is interesting to discuss a quantitative version or stability of this theorem. De Lellis and Müller [6] obtained an L^2 estimate for closed surfaces in $\mathbb{R}^3$. Their result also has

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applications to foliations of asymptotically flat three manifolds by surfaces of prescribed mean curvature [14–16]. Recently, Perez [17] studied the hypersurface case and proved the following theorem for convex hypersurfaces in $\mathbb{R}^{n+1}$:

Theorem 1.1 *([17]) Let Σ be a smooth, closed and connected hypersurface in $\mathbb{R}^{n+1}$, $n \geq 2$ with induced metric g and non-negative Ricci curvature, then*

$$\int_\Sigma \left|A - \frac{\overline{H}}{n} g\right|^2 \leq \frac{n}{n-1} \int_\Sigma \left|A - \frac{H}{n} g\right|^2, \tag{1.1}$$

and equivalently

$$\int_\Sigma \left(H - \overline{H}\right)^2 \leq \frac{n}{n-1} \int_\Sigma \left|A - \frac{H}{n} g\right|^2, \tag{1.2}$$

where A and $H = trA$ denote the second fundamental form and the mean curvature of Σ respectively, $\overline{H} = \frac{1}{Vol_n(\Sigma)} \int_\Sigma H$ is the average of H on Σ. In particular, the above estimates hold for smooth, closed hypersurfaces which are the boundary of a convex set in $\mathbb{R}^{n+1}$.

As pointed out by De Lellis and Topping [7], Perez's theorem holds even for the closed hypersufaces with nonnegative Ricci curvature when the ambient space is Einstein. Indeed a slight modification of the proof of Theorem 1.1 gives the following result (see its proof in [5]).

Theorem 1.2 *Let $(M^{n+1}, \tilde{g})$ be an Einstein manifold, $n \geq 2$. Let Σ be a smooth, closed and connected hypersurface immersed in M with induced metric g. Assume that (Σ, g) has non-negative Ricci curvature. Then the same inequalities as (1.1) and (1.2) hold.*

Later, Zhou and the first author ([5]) studied the rigidity of the equalities in Inequalities (1.1), (1.2), and the correspondging inequalities in Theorem 1.2. They proved that

Theorem 1.3 *([5]) Let M^{n+1} be the Euclidean space $\mathbb{R}^{n+1}$, or the Euclidean semi-sphere $\mathbb{S}^n_+$ or the Hyperbolic space $\mathbb{H}^{n+1}$, $n \geq 2$. Let Σ be a smooth, connected, oriented and closed hypersurface immersed in M^{n+1} with induced metric g. Assume that (Σ, g) has non-negative Ricci curvature. Then,*

$$\int_\Sigma \left|A - \frac{\overline{H}}{n} g\right|^2 = \frac{n}{n-1} \int_\Sigma \left|A - \frac{H}{n} g\right|^2, \tag{1.3}$$

and equivalently

$$\int_\Sigma \left(H - \overline{H}\right)^2 = \frac{n}{n-1} \int_\Sigma \left|A - \frac{H}{n} g\right|^2, \tag{1.4}$$

holds if and only if Σ is a totally umbilical hypersurface, that is, Σ is a distance sphere S^n in M^{n+1}, where $\overline{H} = \frac{1}{Vol_n(\Sigma)} \int_\Sigma H$.

In [5], the authors also studied the general case for hypersurfaces without assumption on convexity (that is, $A \geq 0$, which is equivalent to Ric ≥ 0 when Σ is a closed hypersurface in $\mathbb{R}^{n+1}$). See more details in [5].

Also in [17], Perez showed the constants in Inequalities (1.1) and (1.2) are sharp. In this paper, we generalize his result and prove that the constants in Inequalities (1.1) and (1.2) are sharp when the ambient spaces are other space forms. Precisely, we prove that

Theorem 1.4 *Let $(M_c^{n+1}, \tilde{g})$ be a space form of constant sectional curvature c, $n \geq 2$. Let $C < \sqrt{\frac{n}{n-1}}$ be a positive constant. Then there exists a smooth deformation Σ_t of the geodesic sphere S^n in M_c^{n+1} so that for each hypersurface Σ_t,*

$$\int_{\Sigma_t} \left(H - \overline{H}\right)^2 > C^2 \int_{\Sigma_t} \left|A - \frac{H}{n} g\right|^2, \tag{1.5}$$

and

$$\int_{\Sigma_t} \left|A - \frac{\overline{H}}{n} g\right|^2 > C^2 \int_{\Sigma_t} \left|A - \frac{H}{n} g\right|^2. \tag{1.6}$$

Moreover, Σ_t can be chosen arbitrarily close to S^n so that the Ricci curvature Ric_{Σ_t} of Σ_t is positive.

Theorem 1.4 shows that the constant $\sqrt{\frac{n}{n-1}}$ in Inequalities (1.1) and (1.2) is optimal when the ambient space is a space form.

In this paper, we also deal with higher order mean curvatures and the Newton transformations of the second fundamental form of hypersurfaces (see their definition in Sect. 2. Besides we refer the interested readers to [3,4,18,19,21] and the related references therein).

When (Σ, g) is a hypersurface isometrically immersed in a space form, it can be verified that if the Newton transformations P_r satisfy $P_r = \frac{\mathrm{tr} P_r}{n} g$ on Σ, then Σ has constant rth mean curvature and thus P_r is a constant multiple of the metric g (cf Sect. 5). Moreover, Ros' work [20,21] implies that the round spheres are the only closed embedded hypersurfaces in $\mathbb{R}^{n+1}$ with $P_r = \frac{\mathrm{tr} P_r}{n} g$, $2 \leq r \leq n$. Like the case for the totally umbilical theorem, we may consider a quantitative version or stability of this result. Recently, the first author [3] showed that

Theorem 1.5 *[3] Let $(M_c^{n+1}, \tilde{g})$ be a space form of constant sectional curvature c, $n \geq 2$. Let Σ be a smooth connected closed hypersurface immersed in M_c^{n+1} with induced metric g. Assume that (Σ, g) has nonnegative Ricci curvature, then for $2 \leq r \leq n$,*

$$(n-r)^2 \int_{\Sigma} (s_r - \overline{s_r})^2 \leq n(n-1) \int_{\Sigma} \left|P_r - \frac{(n-r)s_r}{n} g\right|^2, \tag{1.7}$$

and equivalently,

$$\int_{\Sigma} \left|P_r - \frac{(n-r)\overline{s_r}}{n} g\right|^2 \leq n \int_{\Sigma} \left|P_r - \frac{(n-r)s_r}{n} g\right|^2, \tag{1.8}$$

where P_r and $s_r = trP_r$ denote the Newton transformations of the second fundamental form A of Σ and the trace of P_r respectively, $\overline{s_r} = \frac{\int_M s_r dv}{Vol(M)}$ denotes the average of s_r over Σ.

In Sect. 5 of this paper, we consider the optimality of the constants for Inequalities (1.7) and (1.8) and prove that

Theorem 1.6 *Let $(M_c^{n+1}, \tilde{g})$ be a space form of constant sectional curvature c, $n \geq 2$. Let the natural number r $(2 \leq r \leq n-1)$ be given. For any given constants $C_1 < \sqrt{\frac{n(n-1)}{(n-r)^2}}$*

and $C_2 < \sqrt{n}$, there exist smooth deformations $(\Sigma_1)_t$ and $(\Sigma_2)_t$ of the geodesic sphere $\mathbb{S}^n$ in M_c^{n+1} respectively, so that for each t,

$$\int_{(\Sigma_1)_t} (s_r - \overline{s_r})^2 > C_1^2 \int_{(\Sigma_1)_t} \left| P_r - \frac{(n-r)s_r}{n} g \right|^2, \tag{1.9}$$

and

$$\int_{(\Sigma_2)_t} \left| P_r - \frac{(n-r)\overline{s_r}}{n} g \right|^2 > C_2^2 \int_{(\Sigma_2)_t} \left| P_r - \frac{(n-r)s_r}{n} g \right|^2, \tag{1.10}$$

where P_r, s_r and $\overline{s_r}$ are given as in Theorem 1.5.

Moreover, $(\Sigma_1)_t$ and $(\Sigma_2)_t$ can be chosen arbitrarily close to S^n so that the Ricci curvatures $Ric_{(\Sigma_i)_t}$ of $(\Sigma_i)_t$, $i = 1, 2$ are positive.

Observe that by $P_1 = s_1 I - A$ and $s_1 = H$, Theorem 1.4 shows that Theorem 1.6 also holds for $r = 1$.

It is worth of mentioning that there is a parallel phenomenon in the clue of Riemannian geometry. Recall that the Schur's theorem states that the scalar curvature of an Einstein manifold of dimension $n \geq 3$ must be constant. One may consider the stability of Schur's theorem. See some work on this topic in [2,3,7,9–11].

The rest of this paper is organized as follows. In Sect. 2, we give some notation and convention. In particular, we give the definitions of Newton transformation and r-th mean curvatures associated to the second fundamental form. In Sect. 3, we prove the existence of a smooth normal deformation which is needed in the next two sections. In Sect. 4, we give some evolution equations and prove Theorem 1.4. In Sect. 5, we prove Theorem 1.6.

2 Notation and convention

In order to give the definition of high order mean curvatures and the Newton transformation associated with the second fundamental form of a hypersurface, which was introduced by Reilly [18] (cf. [19]), we first recall the definitions of the rth elementary symmetric functions and Newton transformations. Let $\sigma_r : \mathbb{R}^n \to \mathbb{R}$ denote the elementary symmetric function in $\mathbb{R}^n$ given by

$$\sigma_r(x_1, \ldots, x_n) = \sum_{i_1 < \cdots < i_r} x_{i_1} \ldots x_{i_r}, \; 1 \leq r \leq n.$$

Let V be an n-dimensional vector space and $A : V \to V$ be a symmetric linear transformation. Let $e_1, \ldots, e_n$ denote the orthonormal eigenvectors of A and $\eta_1, \ldots, \eta_n$ denote the eigenvalues satisfying $Ae_i = \eta_i e_i$, $i = 1, \ldots, n$ respectively.

Definition 2.1 Define the rth symmetric functions $\sigma_r(A)$, simply denoted by σ_r, associated with A by

$$\sigma_0 = 1, \tag{2.1}$$

$$\sigma_r = \sigma_r(\eta_1, \ldots, \eta_n), \; 1 \leq r \leq n. \tag{2.2}$$

Also define the Newton transformations $P_r : V \to V$, associated with A, $0 \le r \le n$, as

$$P_0 = I \quad \text{(Identidade)}, \tag{2.3}$$

$$\begin{aligned} P_r &= \sum_{j=0}^{r} (-1)^j \sigma_{r-j} A^j \\ &= \sigma_r I - \sigma_{r-1} A + \cdots + (-1)^r A^r, \quad r = 1, \ldots, n. \end{aligned} \tag{2.4}$$

By definition, $P_r = \sigma_r I - AP_{r-1}$, $P_n = 0$. It was proved in [18] that P_r has the following basic properties:

$$(i) \;\; \mathrm{tr}(P_r) = (n-r)\sigma_r; \tag{2.5}$$

$$(ii) \;\; \mathrm{tr}(AP_r) = (r+1)\sigma_{r+1}; \tag{2.6}$$

$$(iii) \;\; \mathrm{tr}(A^2 P_r) = \sigma_1 \sigma_{r+1} - (r+2)\sigma_{r+2}, \tag{2.7}$$

where tr denotes the trace of the corresponding transformation.

Now assume $(M, \tilde{g})$ is an $(n+1)$-dimensional Riemannian manifold, $n \ge 2$. Suppose (Σ, g) is a smooth connected oriented closed hypersurface immersed in $(M, \tilde{g})$ with induced metric g. In this paper, unless otherwise specified, we denote by a " $\sim$" all quantities on $(M, \tilde{g})$, for instance by $\widetilde{\nabla}$ the Levi-Civita connection of $(M, \tilde{g})$. Also we denote for example by ∇, Ric, Δ, the Levi-Civita connection, the Ricci curvature tensor, the Laplacian on (Σ, g) respectively.

Let ν denote the outward unit normal to Σ. The second fundamental form $A = (A_{ij})$ of Σ is defined by

$$\begin{aligned} &A : T_p\Sigma \otimes_s T_p\Sigma \to \mathbb{R}, \\ &A(X, Y) = -\tilde{g}(\widetilde{\nabla}_X Y, \nu), \end{aligned}$$

where $X, Y \in T_p\Sigma$, $p \in \Sigma$.

The second fundamental form A corresponds a $(1, 1)$-tensor on $T_p\Sigma$, which is called the shape operator of Σ and still denoted by A. Hence the shape operator A satisfies that A: $T_p\Sigma \to T_p\Sigma$, $AX = \widetilde{\nabla}_X \nu$, $X \in T_p\Sigma$.

Let η_i, $i = 1, \ldots, n$ denote the principle curvatures of Σ at p, which are the eigenvalues of A at p corresponding the orthonormal eigenvectors $\{e_i\}$, $i = 1 \ldots, n$ respectively.

By Definition 2.1, for the second fundamental form or the shape operator A, we have $s_r = \sigma_r(A)$, the Newton transformations P_r associated with A at p respectively, $0 \le r \le n$ and

Definition 2.2 The rth mean curvature H_r of Σ at p is defined by $s_r = \binom{n}{r} H_r$, $0 \le r \le n$.

For instance, $H_1 = \frac{s_1}{n} = \frac{H}{n}$, where $H = \mathrm{tr}A$ is the mean curvature of Σ. H_n is called the Gauss-Kronecker curvature. When the ambient space M is a space form M_c^{n+1} with constant sectional curvature c,

$$\begin{aligned} &\mathrm{Ric} = (n-1)cI + HA - A^2. \\ &R = \mathrm{trRic} = n(n-1)c + H^2 - |A|^2 = n(n-1)c + 2s_2. \end{aligned}$$

Hence H_2 is, modulo a constant, the scalar curvature of Σ.

In a local coordinate system, $g = (g_{ij})$, its inverse $g^{-1} = (g^{ij})$. Given a smooth function f on Σ, Hess$f = \nabla^2 f$ denotes the Hessian of f on Σ. Throughout this paper, we use Einstein summation convention of summing over repeated indices. We use the *raising and* *lowering indices* to change the type of tensor between a symmetric $(2,0)$ tensor and its corresponding $(1,1)$ tensor. For instance,

$$g_i^j = g_{ik}g^{kj} = \delta_i^j,$$
$$(P_r)_{ij} = g_{ik}(P_r)_j^k.$$

We denote by $\langle\cdot,\cdot\rangle$ the inner product of two smooth tensor fields of the same type on Σ. Given two smooth symmetric $(2,0)$-tensor fields (S_{ij}) and (T_{ij}),

$$\langle(S_{ij}),(T_{ij})\rangle = g^{ij}g^{kl}S_{ik}T_{jl} = S_i^lT_l^i = \langle(S_i^j),(T_i^j)\rangle.$$

We have the notation

$$|(T_{ij})|^2 = \langle(T_{ij}),(T_{ij})\rangle = g^{ij}g^{kl}T_{ik}T_{jl} = T_i^jT_j^i = |(T_i^j)|^2.$$

When there is no confusion, we omit writting the type of tensors, for instance $\langle S,T\rangle$, $|T|^2$. We denote by $\mathrm{tr}T$ and $\mathring{\mathrm{T}}$ the trace of $(2,0)$ tensor T and the traceless part of T: $\mathring{\mathrm{T}} = T - \frac{\mathrm{tr}T}{n}g$ respectively. Then

$$\mathrm{tr}T = g^{ik}T_{ki} = T_i^i.$$

We use the notation: $\partial_tT_{ij} = \frac{\partial}{\partial t}(T_{ij})$, $\partial_tT_i^j = \frac{\partial}{\partial t}(T_i^j)$, which define two tensors $(\partial_tT)_{ij} = \partial_tT_{ij}$ and $(\partial_tT)_i^j = \partial_tT_{ij}$, respectively.

3 A normal deformation

Let $F_0 : \Sigma^n \hookrightarrow (M^{n+1},\tilde{g})$ be a smooth immersion of a closed orientable hypersurface in a Riemannian manifold with the induced metric g. We will prove the short-time existence of the the following initial value problem: a one-parameter family $F(\cdot,t) : \Sigma\times[0,T) \to M^{n+1}$ of hypersurfaces $\Sigma_t = F(\cdot,t)$ satisfies:

$$\begin{cases} \frac{\partial F}{\partial t}(x,t) = f(x)\nu(x,t), & x\in\Sigma, t\in[0,T); \\ F(x,0) \;= F_0(x), & x\in\Sigma. \end{cases} \tag{3.1}$$

where $f(x)$ is a smooth function on Σ and $\nu(x,t)$ denotes the outer unit normal of Σ_t at $F(x,t)$.

The approach is to represent the hypersurface Σ_t as a graph in Fermi coordinates over the initial hypersurface Σ and then to consider the deformation process as a first order PDE equation for the height function.

Let $\phi : \Sigma\times(-\epsilon,\epsilon) \to M^{n+1}$ be an immersion, where ϵ is sufficiently small, by the exponential map

$$\phi(x,h) = \exp_{F_0(x)} h\nu(F_0(x)).$$

ϕ induces a metric $\phi^*\tilde{g}$ on $\Sigma\times(-\epsilon,\epsilon)$ from $(M^{n+1},\tilde{g})$. Let $x_1,\dots,x_n$ denotes a local coordinate on Σ. Then $\{\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n},\frac{\partial}{\partial h}\}$ is a local coordinate frame on $\Sigma\times(-\epsilon,\epsilon)$. By the Gauss lemma, the metric $\phi^*\tilde{g}$ satisfies

$$(\phi^*\tilde{g})_{ih} = \tilde{g}\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial h}\right) = 0, i=1,\dots,n; (\phi^*\tilde{g})_{hh} = \tilde{g}\left(\frac{\partial}{\partial h},\frac{\partial}{\partial h}\right) = 1.$$

Clearly for t fixed, $\phi_t(\Sigma) = \phi(\Sigma,t)$ is a hypersurface in M^{n+1}. If Σ is embedded, ϕ gives the so-called Fermi-coordinates on a tubular neighborhood of $F_0(\Sigma)$.

Given a smooth function $u : \Sigma \to (-\epsilon, \epsilon)$, the map $\psi : \Sigma \to \Sigma \times (-\epsilon, \epsilon)$ by $\psi(x) = (x, u(x))$ is an immersion, i.e., the graph $G(u)$ of u is a hypersurface in $\Sigma \times (-\epsilon, \epsilon)$. A local coordinate frame on $G(u)$ is

$$\frac{\partial \psi}{\partial x_i} = \frac{\partial}{\partial x_i} + \frac{\partial u}{\partial x_i}\frac{\partial}{\partial h}, \quad i = 1, \ldots, n.$$

The unit normal field $\mathbf{n}$ of $G(u)$ is

$$\mathbf{n}(x, u(x)) = \frac{1}{W}\left(\frac{\partial}{\partial h} - (\phi^*\tilde{g})^{ij}\frac{\partial u}{\partial x_i}\frac{\partial}{\partial x_j}\right)(x, u(x)),$$

where $W(x, u(x)) = \sqrt{1 + (\phi^*\tilde{g})^{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}}(x, u(x))$. Hence

$$\langle \mathbf{n}, \frac{\partial}{\partial h}\rangle = \frac{1}{W}, \quad \frac{\partial}{\partial h} = \frac{1}{W}\mathbf{n} + \left(\frac{\partial}{\partial h}\right)^{\top},$$

where $\left(\frac{\partial}{\partial h}\right)^{\top}$ is the projection of $\frac{\partial}{\partial h}$ to the tangent space of $G(u)$ spanned by $\frac{\partial \psi}{\partial x_i}$, $i = 1, \ldots, n$.

Under the above expression, we will prove that

Theorem 3.1 *The initial value problem (3.1) has the unique smooth solution for T sufficiently small.*

Proof Consider the initial value problem of the first order PDE on $\Sigma \times [0, T)$, $T < \epsilon$:

$$\begin{cases} \frac{\partial u}{\partial t}(x, t) = f(x)\sqrt{1 + (\phi^*\tilde{g})^{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}(x, u(x, t))}, \; (x, t) \in \Sigma \times (0, T) \\ u(x, 0) \;\; = 0, \quad x \in \Sigma. \end{cases} \tag{3.2}$$

By the theory of the first order PDE (cf [1] §35–48; or [8], Chapter 3, Section 3), (3.2) exists the unique smooth solution for $t < T$, where T is sufficiently small. Here the closeness of Σ guarantees the global existence of the solution. Also we may choose T sufficiently small so that $u(x, t) \in (-\epsilon, \epsilon)$.

By the solution $u(x, t)$ of (3.2), we may construct a one-parameter family of hypersurfaces, i.e., graphs $G(u_t)$, where $u_t(x) = u(x, t)$, $t \in [0, T)$, parametrised by $\Psi : \Sigma \times [0, T) \to \Sigma \times (-\epsilon, \epsilon)$ satisfying

$$\Psi(x, t) = \Psi_t(x) = (x, u(x, t)).$$

Then

$$\frac{\partial \Psi}{\partial t} = \frac{\partial u}{\partial t}\frac{\partial}{\partial h} = \frac{\partial u}{\partial t}\left(\frac{1}{W}\mathbf{n}\right) + \frac{\partial u}{\partial t}\left(\frac{\partial}{\partial h}\right)^{\top}.$$

So we have

$$\frac{\partial \Psi}{\partial t}(x, t) = f(x)\mathbf{n}(x, t) + \lambda(x, t), \tag{3.3}$$

where $\mathbf{n}(x, t)$ denotes the unit normal of $\Psi_t(\Sigma)$ at $\Psi(x, t)$ and $\lambda = \frac{\partial u}{\partial t}\left(\frac{\partial}{\partial h}\right)^{\top}$ is the projection of $\frac{\partial \Psi}{\partial t}$ to the tangent space of $\Psi_t(\Sigma)$ spanned by $\frac{\partial \Psi}{\partial x_i}$, $i = 1, \ldots, n$.

Let $\alpha : \Sigma \times [0, T) \to \Sigma$ be a smooth one-parameter family of diffeomorphisms of Σ satisfying

$$\frac{\partial \alpha}{\partial t} = -\Psi_t^*\lambda, \quad \alpha_0 = Id \quad \text{(Identity)} \tag{3.4}$$

Integrating (3.4) directly, we obtain the unique smooth solution. Define $\Phi(x,t) = \Psi(\alpha(x,t),t)$. Then

$$\begin{aligned}\frac{\partial \Phi}{\partial t}(x,t) &= \Psi_*\left(\frac{\partial \alpha}{\partial t}\right)(\alpha(x,t),t) + \left(\frac{\partial}{\partial t}\Psi(\cdot,t)\right)(\alpha(x,t),t)\\ &= -\lambda\Psi(\alpha(x,t),t) + f(x)\nu(\alpha(x,t),t) + \lambda\Phi(x,t)\\ &= f(x)\mathbf{n}(x,t).\end{aligned}$$

Finally note the exponential map ϕ is a local isometry. Define $F:\Sigma\times[0,T)\to M^{n+1}$ by $F=\phi\circ\Phi$. Then $F(x,t)$ is just the solution of Problem (3.1). □

4 Evolution equations of geometric quantities and proof of Theorem 1.4

In this section, we will prove Theorem 1.4. Although there is a unified proof of it and Theorem 1.6, we prefer to give an independent proof of Theorem 1.4. One reason is that the evolution equations for mean curvature and the shape operator are much more simple than the ones for r-th mean curvatures and the Newton transformations P_r, $2\le r\le n$. We use the method by Perez [17] in proving Theorem 1.1.

Let $F_0:\Sigma^n\hookrightarrow(M^{n+1},\tilde g)$ be a smooth immersion of a closed orientable hypersurface in a Riemannian manifold with the induced metric g. Consider the normal deformation of hypersurfaces according to the equation:

$$\begin{cases}\frac{\partial F}{\partial t}(x,t) = f(x,t)\nu(x,t), & x\in\Sigma, t\in[0,T);\\ F(x,0) = F_0(x), & x\in\Sigma.\end{cases} \tag{4.1}$$

where $f(x,t)$ is a smooth function on Σ and $\nu(x,t)$ denotes the outer unit normal of $\Sigma_t = F(\Sigma,t)$ at $F(x,t)$.

If the solution of (4.1) exists, the following basic evolution equations holds under (4.1) (cf [12,13]):

Proposition 4.1 *For any solution of* (4.1)*, it holds that*

$$\partial_t g_{ij} = 2fA_{ij}, \tag{4.2}$$

$$\partial_t g^{ij} = -2fA^{ij}, \tag{4.3}$$

$$\partial_t g_i^j = 0 \tag{4.4}$$

$$\partial_t(dvol) = fHdvol, \tag{4.5}$$

$$\partial_t\nu = -\nabla f, \tag{4.6}$$

$$\partial_t A_{ij} = -(Hess f)_{ij} + f(A^2)_{ij} - f\widetilde{R}_{0i0j}, \tag{4.7}$$

$$\partial_t H = -\Delta f - |A|^2 f - f\widetilde{Ric}(\nu,\nu). \tag{4.8}$$

In particular, when the ambient space is the space form M_c^{n+1} with the sectional curvature c,

$$\partial_t A_{ij} = -(Hess f)_{ij} + f(A^2)_{ij} - cfg_{ij}, \tag{4.9}$$

$$\partial_t H = -\Delta f - |A|^2 f - ncf. \tag{4.10}$$

Here and thereafter, for simplicity, we drop the t subscript wherever it would not lead to confusion. For instance, g and A denote the induced metric $g_t = F_t^*(\tilde g)$ and the second fundamental form of (Σ_t, g_t) respectively. When the ambient space is the space form M_c^{n+1}, Proposition 4.1 yields

Proposition 4.2

$$\partial_t A_i^j = -(Hess f + f A^2 + cfg)_i^j. \tag{4.11}$$

$$\partial_t |A|^2 = -2\langle Hess f, A\rangle - 2f trA^3 - 2cfH. \tag{4.12}$$

$$\partial_t |\mathring{A}|^2 = -2\langle Hess f, \mathring{A}\rangle - 2f\langle A^2, \mathring{A}\rangle, \tag{4.13}$$

where $\mathring{A} = A - \frac{trA}{n} g$.

Proof

$$\begin{aligned}
\partial_t A_i^j &= \partial_t (A_{ik} g^{kj}) \\
&= (\partial_t A_{ik}) g^{kj} + A_{ik} \partial_t g^{kj} \\
&= [-(\text{Hess} f)_{ik} + f(A^2)_{ik} - cfg_{ik}] g^{kj} - 2f A_{ik} A^{kj} \\
&= -(\text{Hess} f)_i^j - f(A^2)_i^j - cfg_i^j \\
\partial_t |A|^2 &= \partial_t (A_i^j A_j^i) \\
&= (\partial_t A_i^j) A_j^i + A_i^j (\partial_t A_j^i) \\
&= -(\text{Hess} f + fA^2 + cfg)_i^j A_j^i - A_i^j (\text{Hess} f + fA^2 + cfg)_j^i \\
&= -2\langle \text{Hess} f, A\rangle - 2f \text{tr} A^3 - 2cfH. \\
\partial_t |\mathring{A}|^2 &= \partial_t \left(|A|^2 - \frac{H^2}{n} \right) \\
&= \partial_t (|A|^2) - \frac{2H}{n} \partial_t H \\
&= -2\langle \text{Hess} f, A\rangle - 2f \text{tr} A^3 - 2cfH + \frac{2H}{n} (\Delta f + |A|^2 f + ncf) \\
&= -2\langle \text{Hess} f, A\rangle + \frac{2H}{n} \langle \text{Hess} f, g\rangle - 2f\langle A^2, A\rangle + \frac{2H}{n} \langle A^2, g\rangle \\
&= -2\langle \text{Hess} f, \mathring{A}\rangle - 2f\langle A^2, \mathring{A}\rangle.
\end{aligned}$$

□

In the rest of this section, we will consider the case that the ambient space is the space form M_c^{n+1} and $f(x,t) = f(x)$, that is, the normal deformation (3.1). Assume that Σ is a closed totally umbilical hypersurface in M_c^{n+1}. It is well known that Σ must be a geodesic sphere, i.e, distance sphere $\mathbb{S}^n(a)$, where a denotes its geodesic radius. Assume $F(x,t) : \Sigma \times [0,T) \to M_c^{n+1}$ be the solution of the normal deformation (3.1). We consider the functional $\mathcal{F} : \Sigma_t = F(\Sigma, t) \to \mathbb{R}$ given by

$$\mathcal{F}(\Sigma_t) = C^2 \int_\Sigma |\mathring{A}|^2 - \int_\Sigma \left(H - \overline{H}\right)^2, \tag{4.14}$$

where C is a constant and the subscripts t of $\mathring{A}$, H and $\overline{H}$ are omitted.

Obviously, $\mathcal{F}(\Sigma) = 0$. Next we obtain the first variation of $\mathcal{F}$ at $t = 0$ as follows.

Proposition 4.3 *If Σ is totally umbilical, then $\frac{d}{dt}\mathcal{F}(\Sigma_t)|_{t=0} = 0$.*

Proof

$$\frac{d}{dt}\mathcal{F}(\Sigma_t) = C^2 \int\limits_{\Sigma} \partial_t |\mathring{A}|^2 dvol + C^2 \int\limits_{\Sigma} |\mathring{A}|^2 \partial_t (dvol) - \int\limits_{\Sigma} \partial_t (H - \overline{H})^2 dvol - \int\limits_{\Sigma} \left(H - \overline{H}\right)^2 \partial_t (dvol).$$

At $t = 0$,

$$\mathring{A} = 0. \tag{4.15}$$

By (??),

$$H = \overline{H}. \tag{4.16}$$

$$\partial_t (H - \overline{H})^2|_{t=0} = 2(H - \overline{H})\partial_t (H - \overline{H})|_{t=0} = 0. \tag{4.17}$$

By (4.13), or directly

$$\begin{aligned} \partial_t |\mathring{A}|^2|_{t=0} &= \partial_t \left(\mathring{A}_i^j \mathring{A}_j^i\right)|_{t=0} \\ &= \left(\partial_t \mathring{A}_i^j\right) \mathring{A}_j^i|_{t=0} + \mathring{A}_i^j \left(\partial_t \mathring{A}_j^i\right)|_{t=0} \\ &= 0. \end{aligned} \tag{4.18}$$

Hence $\frac{d}{dt}\mathcal{F}(\Sigma_t)|_{t=0} = 0$. □

Furthermore we discuss the second variation of $\mathcal{F}$. The straightforward computation implies the following conclusion.

Proposition 4.4 *Suppose $\varphi(\cdot, t) : \Sigma \times (-\epsilon, \epsilon) \to \mathbb{R}$ is a smooth function. If $\varphi|_{t=0} = 0$ and $\partial_t \varphi|_{t=0} = 0$, then*

$$\left(\frac{d^2}{dt^2} \int\limits_{\Sigma} \varphi\right)(0) = \int\limits_{\Sigma} \partial_t^2 \varphi|_{t=0}. \tag{4.19}$$

Take $\phi = |\mathring{A}|^2$ and $(H - \overline{H})^2$ in Proposition 4.4 respectively. By (4.15), (4.16), (4.17) and (4.18), it holds that

$$\frac{d^2}{dt^2} F(\Sigma_t)|_{t=0} = C^2 \int\limits_{\Sigma} \partial_t^2 |\mathring{A}|^2|_{t=0} - \int\limits_{\Sigma} \partial_t^2 (H - \overline{H})^2|_{t=0} \tag{4.20}$$

Now we will calculate the right hand of (4.20).

$$\begin{aligned} \partial_t \mathring{A}_i^j &= \partial_t \left(A_i^j - \frac{H}{n} g_i^j\right) \\ &= \partial_t A_i^j - \frac{1}{n}(\partial_t H) g_i^j - \frac{H}{n} \partial_t g_i^j \\ &= -[\mathrm{Hess} f + f A^2 + f c g]_i^j + \frac{1}{n}(\Delta f + |A|^2 f + n c f) g_i^j \end{aligned}$$

$$= -(\text{Hess} f)_i^j + \frac{1}{n}(\Delta f) g_i^j - f(A^2)_i^j + \frac{1}{n}|A|^2 f g_i^j$$
$$= -(\text{Hess} f)_i^j + \frac{1}{n}(\Delta f) g_i^j - f(A\mathring{A})_i^j + \frac{1}{n}|\mathring{A}|^2 f g_i^j - \frac{1}{n} H f \mathring{A}_i^j. \tag{4.21}$$

In the last equality of (4.21), we used the identity

$$A^2 - \frac{1}{n}|A|^2 g = A\mathring{A} + \frac{H}{n} A - \frac{1}{n}\left(|\mathring{A}|^2 + \frac{H^2}{n}\right) g = f A\mathring{A} - \frac{1}{n}|\mathring{A}|^2 + \frac{H}{n}\mathring{A}.$$

Note $\mathring{A}|_{t=0} = 0$. By (4.21),

$$\partial_t \mathring{A}_i^j|_{t=0} = -(\text{Hess} f)_i^j|_{t=0} + \frac{1}{n}(\Delta f) g_i^j|_{t=0}$$
$$= -(\mathring{\text{Hess}} f)_i^j|_{t=0},$$

and

$$\partial_t^2 |\mathring{A}|^2|_{t=0} = \partial_t \left[\partial_t \left(\mathring{A}_i^j \mathring{A}_j^i\right)\right]|_{t=0}$$
$$= \left[\left(\partial_t^2 \mathring{A}_i^j\right)\mathring{A}_j^i + 2\left(\partial_t \mathring{A}_i^j\right)\left(\partial_t \mathring{A}_j^i\right) + \mathring{A}_i^j\left(\partial_t^2 \mathring{A}_j^i\right)\right]|_{t=0}$$
$$= 2\left(\partial_t \mathring{A}_i^j|_{t=0}\right)\left(\partial_t \mathring{A}_j^i|_{t=0}\right)$$
$$= 2\left(\mathring{\text{Hess}} f\right)_i^j|_{t=0}\left(\mathring{\text{Hess}} f\right)_j^i|_{t=0}$$
$$= 2|\mathring{\text{Hess}} f|^2|_{t=0} \tag{4.22}$$

In order to calculate the second term of the right side of (4.20), we need the following proposition, which was proved in [17].

Proposition 4.5 *[17] Suppose $\varphi(\cdot, t) : \Sigma \times (-\epsilon, \epsilon) \to \mathbb{R}$ be a smooth function. Let $\overline{\varphi}(t) = \frac{1}{vol(\Sigma_t)} \int_\Sigma \varphi(t) dvol(\Sigma_t)$ be average of $\varphi(t)$ on Σ_t. Then*

$$\frac{d}{dt}\overline{\varphi} = \overline{\partial_t \varphi} + \overline{f H(\varphi - \overline{\varphi})}. \tag{4.23}$$

For the completeness of proof, we include its proof here.

Proof

$$\partial_t \overline{\varphi} = \left[\partial_t \left(\frac{1}{\text{vol}(\Sigma_t)}\right)\right] \int_\Sigma \varphi dvol + \frac{1}{\text{vol}(\Sigma_t)} \partial_t \left(\int_\Sigma \varphi dvol\right)$$
$$= -\frac{1}{\text{vol}(\Sigma_t)^2}\left[\partial_t \left(\int_\Sigma dvol\right)\right] \int_\Sigma \varphi dvol$$
$$+ \frac{1}{\text{vol}(\Sigma_t)} \int_\Sigma \partial_t \varphi dvol + \frac{1}{\text{vol}(\Sigma_t)} \int_\Sigma \varphi \partial_t (dvol)$$

$$= -\frac{1}{\mathrm{vol}(\Sigma_t)}\left(\int_\Sigma fH\overline{\varphi}dvol\right) + \overline{\partial_t\varphi} + \frac{1}{\mathrm{vol}(\Sigma_t)}\left(\int_\Sigma fH\varphi dvol\right)$$
$$= \overline{\partial_t\varphi} + \overline{fH(\varphi - \overline{\varphi})}.$$

□

Taking $\varphi = H$ in Proposition 4.5, we have

$$\partial_t\overline{H} = \overline{\partial_t H} + \overline{fH(H - \overline{H})}. \tag{4.24}$$

At $t = 0$, by (4.16), (4.8) and $|A| = \frac{H^2}{n}$, it holds that

$$\begin{aligned}
\partial_t(H - \overline{H})|_{t=0} &= (\partial_t H - \partial_t\overline{H})|_{t=0} \\
&= (\partial_t H - \overline{\partial_t H} - \overline{fH(H - \overline{H})})|_{t=0} \\
&= (\partial_t H - \overline{\partial_t H})|_{t=0} \\
&= -\Delta f - |A|^2 f - ncf + \overline{\Delta f + |A|^2 f + ncf} \\
&= -\Delta f - \frac{H^2}{n}f - ncf + \overline{\Delta f + \frac{H^2}{n}f + ncf} \\
&= -\Delta f - n\omega f + \overline{\Delta f + n\omega f},
\end{aligned}$$

where $\omega = \frac{H^2}{n^2} + c$. So

$$\begin{aligned}
\partial_t^2(H - \overline{H})^2|_{t=0} &= 2[\partial_t(H - \overline{H})]^2 + 2(H - \overline{H})\partial_t^2(H - \overline{H}) \\
&= 2[\partial_t(H - \overline{H})]^2| \\
&= 2(\Delta f + n\omega f - \overline{\Delta f + n\omega f})^2
\end{aligned} \tag{4.25}$$

Thus (4.20) together with (4.22) and (4.25) yields that

Proposition 4.6 *Let Σ is a totally umbilical hypersurface in the space form. It holds that*

$$\frac{d^2}{dt^2}\mathcal{F}(\Sigma_t)|_{t=0} = 2C^2\int_\Sigma |\mathring{H}ess f|^2(0) - 2\int_\Sigma (\Delta f + n\omega f - \overline{\Delta f + n\omega f})^2. \tag{4.26}$$

In the following, we will show that

Theorem 4.1 *Let M_c^{n+1} be a space form of the constant sectional curvature c. Then there exists a closed totally umbilical hypersurface Σ in M and a deformation $F(\cdot, t)$ of Σ such that*

$$\frac{d^2}{dt^2}\mathcal{F}(\Sigma_t)|_{t=0} < 0,$$

where $\mathcal{F}(\Sigma_t)$ is given by (4.14) with constant $C < \sqrt{\frac{n}{n-1}}$.

Proof We first consider the case of the simply connected space forms M_c^{n+1}, i.e., the Euclidean space $\mathbb{R}^{n+1}, c = 0$, the Euclidean sphere $\mathbb{S}^{n+1}, c > 0$, and the hyperbolic space $\mathbb{H}^{n+1}, c < 0$ respectively. Here for convenience we take the Poincaré model for $\mathbb{H}^{n+1}$. Given an M_c^{n+1}, its rotationally symmetric metric is denoted by

$$\tilde{g} = dr^2 + sn_c^2(r)\eta,$$

where η denotes the metric of the unit Euclidean sphere $\mathbb{S}^n$, r denotes the distance under the metric $\tilde{g}$ to the pole o and $sn_c(r)$ is a function given by

$$sn_c(r) = \begin{cases} r, & \text{if } c = 0; \\ \frac{\sin(\sqrt{c}r)}{\sqrt{c}}, & \text{if } c > 0; \\ \frac{\sinh(\sqrt{|c|}r)}{\sqrt{|c|}}, & \text{if } c < 0. \end{cases}$$

Now fix a number $a > 0$ (in the case of $\mathbb{S}^{n+1}$, $0 < a < \frac{\pi}{\sqrt{c}}$) and choose Σ as the geodesic sphere $\mathbb{S}^n(a)$ in M with the geodesic radius a centered at o. It is well known that Σ is totally umbilical. On the other hand, Σ has the induced metric $g = sn_c^2(a)\eta$. The metric g is the metric of the round sphere with the radius $sn_c(a)$ and so the Ricci curvatue of Σ is

$$Ric_\Sigma(\nabla f, \nabla f) = \frac{n-1}{sn_c^2(a)}|\nabla f|^2.$$

Recalling the Bochner formula

$$\frac{1}{2}\Delta|\nabla f|^2 = |\mathrm{Hess} f|^2 + \mathrm{Ric}^\Sigma(\nabla f, \nabla f) + \langle \nabla f, \nabla(\Delta f)\rangle ,$$

and integrating it, by the Stokes' formula, we have

$$\begin{aligned} \int_\Sigma |\mathrm{Hess} f|^2 &= \int_\Sigma (\Delta f)^2 - \int_\Sigma \mathrm{Ric}_\Sigma(\nabla f, \nabla f) \\ &= \int_\Sigma (\Delta f)^2 - \frac{n-1}{sn_c^2(a)} \int_\Sigma |\nabla f|^2 \\ \int_\Sigma |\mathring{\mathrm{H}}\mathrm{ess} f|^2 &= \int_\Sigma |\mathrm{Hess} f|^2 - \frac{1}{n}\int_\Sigma (\Delta f)^2 \\ &= \frac{n-1}{n}\int_\Sigma (\Delta f)^2 - \frac{n-1}{sn_c^2(a)}\int_\Sigma |\nabla f|^2 \\ &= \frac{n-1}{n}\int_\Sigma (\Delta f)^2 + \frac{n-1}{sn_c^2(a)}\int_\Sigma f\Delta f. \end{aligned}$$

Note that the closeness of Σ implies the average $\overline{\Delta f} = 0$. Assume that f satisfies $\int_\Sigma f = 0$ (such f will be chosen later). Then

$$\overline{\Delta f + n\omega f} = \overline{\Delta f} + n\omega\overline{f} = 0$$

and so

$$\begin{aligned} \int_\Sigma \left(\Delta f + n\omega f - \overline{\Delta f + n\omega f}\right)^2 &= \int_\Sigma (\Delta f + n\omega f)^2 \\ &= \int_\Sigma (\Delta f)^2 + 2n\omega \int_\Sigma f\Delta f + n^2\omega^2 \int_\Sigma f^2. \end{aligned}$$

Thus

$$
\begin{aligned}
&\frac{1}{2}\frac{d^2}{dt^2}\mathcal{F}(\Sigma_t)|_{t=0}\\
&\quad=\frac{C^2(n-1)}{n}\int\limits_{\Sigma}(\Delta f)^2+\frac{C^2(n-1)}{sn_c^2(r)}\int\limits_{\Sigma}f\Delta f\\
&\qquad-\int\limits_{\Sigma}(\Delta f)^2-2n\omega\int\limits_{\Sigma}f\Delta f-n^2\omega^2\int\limits_{\Sigma}f^2\\
&\quad=\left[\frac{C^2(n-1)}{n}-1\right]\int\limits_{\Sigma}(\Delta f)^2+\left[\frac{C^2(n-1)}{sn_c^2(a)}-2n\omega\right]\int\limits_{\Sigma}(f\Delta f)-n^2\omega^2\int\limits_{\Sigma}f^2
\end{aligned}
$$

Now we choose f to be an eigenfunction of the Laplacian on (Σ, g) corresponding to the nonzero eigenvalue $\xi(k)$, that is,

$$
\Delta_g f=-\xi(k)f,\quad \int\limits_{\Sigma}f=0.
$$

It is known that on Σ,

$$
\Delta_g=\frac{1}{sn_c^2(a)}\Delta_\eta.
$$

Hence the nonzero eigenvalues $\xi(k)$ are

$$
\xi(k)=\frac{k(k+n-1)}{sn_c^2(a)}, k=1,2,\ldots
$$

The sequence $\xi(k)$ increases and diverges to the $+\infty$ as k tends to $+\infty$. For such f,

$$
\begin{aligned}
&\frac{1}{2}\frac{d^2}{dt^2}\mathcal{F}(\Sigma_t)|_{t=0}\\
&\quad=\left(\frac{C^2(n-1)}{n}-1\right)\int\limits_{\Sigma}(\Delta f)^2+\left(\frac{C^2(n-1)}{sn_c^2(a)}-2n\omega\right)\int\limits_{\Sigma}(f\Delta f)-n^2\omega^2\int\limits_{\Sigma}f^2\\
&\quad=\left(\frac{C^2(n-1)}{n}-1\right)\xi(k)^2\int\limits_{\Sigma}f^2-\left(\frac{C^2(n-1)}{sn_c^2(a)}-2n\omega\right)\xi(k)\int\limits_{\Sigma}f^2-n^2\omega^2\int\limits_{\Sigma}f^2\\
&\quad=\left[\left(\frac{C^2(n-1)}{n}-1\right)\xi(k)^2-\left(\frac{C^2(n-1)}{sn_c^2(a)}-2n\omega\right)\xi(k)-n^2\omega^2\right]\int\limits_{\Sigma}f^2
\end{aligned}
$$

When $C<\sqrt{\frac{n-1}{n}}$, the coefficient of $\xi^2(k)$ in the last equality is negative. Hence, if ξ big enough, the quadratic polynomial $\frac{1}{2}\frac{d^2}{dt^2}\mathcal{F}(\Sigma_t)|_{t=0}$ is negative. By the property of $\xi(k)$, there exists a k_0 sufficiently large so that $k\geq k_0$, $\frac{1}{2}\frac{d^2}{dt^2}\mathcal{F}(\Sigma_t)|_{t=0}<0$. This is the case of the simply connected M_c^{n+1}. Observe that the geodesic radius a of the geodesic sphere Σ can be arbitrarily chosen only if it makes sense.

If M_c^{n+1} is not simply connected, we consider the universal covering map $\pi : \tilde{M} \to M_c^{n+1}$. It is known that π is a local isometry. Let $\Omega \subset \tilde{M}$ be a neighborhood of the pole $o \in \tilde{M}$ such that $\pi : \Omega \to \pi(\Omega) \subset M$ is an isometry. In Ω, by the conclusion of the simply connected space form, there exists a geodesic sphere $\tilde{\Sigma} \subset \Omega$ (with small geodesic radius) and a deformation $\tilde{F}(\cdot, t) : \tilde{\Sigma} \times (-\epsilon, \epsilon) \to \mathbb{R}$, which has the conclusion of the theorem. Take $\Sigma = \pi(\tilde{\Sigma})$ and the deformation $F(\cdot, t) = \tilde{F}(\pi^{-1}(\cdot), t)$. Then Σ and $\mathcal{F}$ satisfy the theorem. □

The proof of Theorem 1.4 For the deformation $F(x, t)$ given in Theorem 4.1 with $\Sigma_0 = \mathbb{S}^n$, the functional $\mathcal{F}$ satisfies

$$\mathcal{F}(\Sigma_t)|_{t=0} = \frac{d}{dt}\mathcal{F}(\Sigma_t)|_{t=0} = 0, \quad \frac{d^2}{dt^2}\mathcal{F}(\Sigma_t)|_{t=0} < 0. \tag{4.27}$$

(4.27) implies that $\mathcal{F}(\Sigma_t) < 0$ for t sufficiently small, that is, it holds on Σ_t:

$$\int_{\Sigma_t} \left(H - \overline{H}\right)^2 > C^2 \int_{\Sigma_t} |A - \frac{H}{n} g|^2. \tag{4.28}$$

By (4.28) and the identity:

$$|A - \frac{\overline{H}}{n} g|^2 = |A - \frac{H}{n} g|^2 + \frac{1}{n}\left(H - \overline{H}\right)^2, \tag{4.29}$$

we have that for $C < \sqrt{\frac{n}{n-1}}$,

$$\begin{aligned} \int_{\Sigma_t} |A - \frac{\overline{H}}{n} g|^2 &> \left(1 + \frac{C^2}{n}\right) \int_{\Sigma_t} |A - \frac{H}{n} g|^2 \\ &> C^2 \int_{\Sigma_t} |A - \frac{H}{n} g|^2. \end{aligned} \tag{4.30}$$

Since Σ_t is arbitrarily close to $\mathbb{S}^n(a)$, the Ricci curvature $\mathrm{Ric}_{\Sigma_t} > 0$. So we complete the proof of theorem. □

5 Proof of Theorem 1.6

In this section, we first give the needed evolution equation of s_r (roughly r-th mean curvatures) under the general normal deform. Next we prove Theorem 1.6. For $r \geq 2$, instead of calculating the complicated evolution equations of P_r, we compute directly the corresponding values at $t = 0$ by using the fact that $\Sigma = \Sigma_0$ is totally umbilical (see (5.29)).

Consider the normal deformation $F(x, t)$ of hypersurfaces in (4.1). Recall a result proved by Reilly [18].

Proposition 5.1 *[18] Let $B = B(t)$ be a smooth one-parameter family of diagonalizable linear transformation of the vector space V, σ_r the symmetric functions of the eigenvalues of B and Q_r the Newton transformation with respect to B. Then for $r = 0, 1, \ldots, n$ we have*

$$\partial_t \sigma_{r+1} = tr((\partial_t B) Q_r).$$

Applying Proposition 5.1 to the shape operator A, by (4.11), we have

$$\begin{aligned}\frac{\partial}{\partial t}(s_r) &= \mathrm{tr}((\partial_t A)P_{r-1})\\ &= -\mathrm{tr}(P_{r-1}\mathrm{Hess} f) - f\,\mathrm{tr}(P_{r-1}A^2) - cf\,\mathrm{tr}P_{r-1}.\end{aligned}$$

By (2.5) and (2.7),

$$\frac{\partial}{\partial t}(s_r) = -\mathrm{tr}(P_{r-1}\mathrm{Hess} f) - f\left(s_1 s_r - (r+1)s_{r+1}\right) - c(n-r+1)s_{r-1}.$$

So it holds that

Corollary 5.1 *[18] Under* (4.1)*,*

$$\frac{\partial}{\partial t}(s_r) = -tr(P_{r-1}Hess f) - f\,tr(P_{r-1}A^2) - cf\,tr P_{r-1} \tag{5.1}$$

$$= -tr(P_{r-1}Hess f) - f\left(s_1 s_r - (r+1)s_{r+1}\right) - c(n-r+1)s_{r-1}. \tag{5.2}$$

To show Theorem 1.6, we will use the same approach as in the proof of Theorem 1.4. In the rest of this section, we assume that $F(x,t): \Sigma \times [0,T) \to M_c^{n+1}$ is the solution of the normal deformation (3.1) and Σ is a totally umbilical hypersurface. Define the functional $\mathcal{G}: \Sigma_t = F(\Sigma, t) \to \mathbb{R}$ given by

$$\mathcal{G}(\Sigma_t) = C^2 \int\limits_{\Sigma} |\mathring{\mathrm{P}}_r|^2 - \int\limits_{\Sigma} (s_r - \overline{s_r})^2\,, \tag{5.3}$$

where C is a constant and the subscripts t of $\mathring{\mathrm{P}}_r$, s_r and $\overline{s_r}$ are omitted.

First we need some combinatorial identities.

Proposition 5.2

$$\sum_{i=0}^{r}(-1)^i\binom{n}{r-i} = \binom{n-1}{r} \tag{5.4}$$

$$\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1} \tag{5.5}$$

$$n\binom{n-1}{r} = (n-r)\binom{n}{r} \tag{5.6}$$

$$\sum_{i=0}^{r}(-1)^i\binom{n}{r-i}i = -\binom{n-2}{r-1} \tag{5.7}$$

(5.4), (5.5) and (5.6) are well known. Since we couldn't find the adequate reference for (5.7), for the completeness of the proof, we prove (5.7) here.

Proof

$$
\begin{aligned}
&\sum_{i=0}^{r}(-1)^i\binom{n}{r-i}i \\
&= -\sum_{i=0}^{r}(-1)^i\binom{n}{r-i}(r-i)+\sum_{i=0}^{r}(-1)^i\binom{n}{r-i}r \\
&= -\sum_{i=0}^{r-1}(-1)^i\frac{n(n-1)\ldots[n-(r-i)+1]}{(r-i)!}(r-i)+r\sum_{i=0}^{r}(-1)^i\binom{n}{r-i} \\
&= -n\sum_{i=0}^{r-1}(-1)^i\frac{(n-1)\ldots[n-1-(r-i-1)+1]}{(r-i-1)!}+r\binom{n-1}{r} \\
&= -n\sum_{i=0}^{r-1}(-1)^i\binom{n-1}{r-i-1}+r\binom{n-1}{r} \\
&= -n\binom{n-2}{r-1}+r\binom{n-1}{r} \\
&= -n\frac{n-r}{n-1}\binom{n-1}{r-1}+(n-r)\binom{n-1}{r-1} \\
&= -\frac{n-r}{n-1}\binom{n-1}{r-1}.
\end{aligned}
\tag{5.8}
$$

In the verification of (5.8), we used (5.4). By (5.6) and (5.8),

$$
\sum_{i=0}^{r}(-1)^i\binom{n}{r-i}i=-\binom{n-2}{r-1}. \tag{5.9}
$$

□

Recall that Σ is a closed totally umbilical hypersurface, i.e., a geodesic sphere $\mathbb{S}^n(a)$ with the geodesic radius a in the space form M_c^{n+1}. Denote by λ the principle curvatures of Σ. λ is a constant. So for $\Sigma=\Sigma_0$,

$$
H=n\lambda, \tag{5.10}
$$

$$
A=\frac{H}{n}I=\lambda I, \tag{5.11}
$$

$$
A^k=\lambda^k I, \tag{5.12}
$$

$$
s_r=\sum_{i_1<\cdots<i_r}\lambda_{i_1}\ldots\lambda_{i_r}=\sum_{i_1<\cdots<i_r}\lambda^r=\binom{n}{r}\lambda^r, \tag{5.13}
$$

$$
\overline{s_r}=s_r. \tag{5.14}
$$

Moreover, by the definition (2.4) of P_r and (5.4), for $1 \le r \le n$,

$$\begin{aligned} P_r &= \sum_{j=0}^{r}(-1)^j s_{r-j} A^j \\ &= \sum_{j=0}^{r}(-1)^j \binom{n}{r-j} \lambda^{r-j}\lambda^j I \\ &= \lambda^r \binom{n-1}{r} I. \end{aligned} \tag{5.15}$$

By (2.5), (5.6), (5.13), and (5.15).

$$\mathring{P}_r = 0. \tag{5.16}$$

So (5.16) and (5.14) imply that $\mathcal{G}(\Sigma_t)|_{t=0} = 0$. The first variation of $\mathcal{G}$ at $t = 0$ can be calculated as follows.

$$\begin{aligned} \partial_t(|\mathring{P}_r|^2)|_{t=0} &= \partial_t[(\mathring{P}_r)_i^j(\mathring{P}_r)_j^i]|_{t=0} \\ &= [\partial_t(\mathring{P}_r)_i^j](\mathring{P}_r)_j^i|_{t=0} + (\mathring{P}_r)_i^j\partial_t[(\mathring{P}_r)_j^i]|_{t=0} \\ &= 0. \end{aligned} \tag{5.17}$$

$$\partial_t(s_r - \overline{s_r})^2|_{t=0} = 2(s_r - \overline{s_r})\partial_t(s_r - \overline{s_r})|_{t=0} = 0. \tag{5.18}$$

So

$$\begin{aligned} \frac{d}{dt}\mathcal{G}(\Sigma_t)|_{t=0} &= \Bigg[C^2\int_\Sigma |\mathring{P}_r|^2\partial_t(dvol) + C^2\int_\Sigma \partial_t(|\mathring{P}_r|^2)dvol \\ &\quad - \int_\Sigma \partial_t(s_r - \overline{s_r})^2 dvol - \int_\Sigma (s_r - \overline{s_r})^2\partial_t(dvol)\Bigg]|_{t=0} \\ &= 0. \end{aligned}$$

We obtain that

Proposition 5.3 *For* Σ, $\frac{d}{dt}\mathcal{G}(\Sigma_t)|_{t=0} = 0.$

Next we calculate the second variation of $\mathcal{G}(\Sigma_t)$ at $t = 0$. By (5.14), (5.16), (5.17) and (5.18), Proposition 4.5 implies that

$$\frac{d^2}{dt^2}\mathcal{G}(\Sigma_t)|_{t=0} = C^2\int_\Sigma \partial_t^2|\mathring{P}_r|^2|_{t=0} - \int_\Sigma \partial_t^2(s_r - \overline{s_r})^2|_{t=0}. \tag{5.19}$$

We have

$$\begin{aligned} \partial_t(|\mathring{P}_r|^2) &= \left[\partial_t(\mathring{P}_r)_i^j\right](\mathring{P}_r)_j^i + (\mathring{P}_r)_i^j\left[\partial_t(\mathring{P}_r)_i^j\right], \\ \partial_t^2(|\mathring{P}_r|^2) &= \left[\partial_t\partial_t(\mathring{P}_r)_i^j\right](\mathring{P}_r)_j^i + 2\left[\partial_t(\mathring{P}_r)_i^j\right]\left[\partial_t(\mathring{P}_r)_j^i\right] + (\mathring{P}_r)_i^j\left[\partial_t\partial_t(\mathring{P}_r)_i^j\right]. \end{aligned}$$

So

$$\partial_t^2(|\mathring{P}_r|^2)|_{t=0} = 2\left[\partial_t(\mathring{P}_r)_i^j\right]\left[\partial_t(\mathring{P}_r)_j^i\right]_{t=0} = 2|\partial_t\mathring{P}_r|^2_{t=0}, \tag{5.20}$$

where the (1, 1)-tensor $\partial_t P_r$ is defined by $(\partial_t P_r)_i^j := \partial_t[(P_r)_i^j]$. On the other hand, we have

$$\begin{aligned}
&\partial_t(s_r - \overline{s_r})^2 = 2(s_r - \overline{s_r})\partial_t(s_r - \overline{s_r}).\\
&\partial_t^2(s_r - \overline{s_r})^2 = 2(s_r - \overline{s_r})\partial_t^2(s_r - \overline{s_r}) + 2[\partial_t(s_r - \overline{s_r})]^2.\\
&\partial_t^2(s_r - \overline{s_r})^2(0) = 2[\partial_t(s_r - \overline{s_r})]^2(0).
\end{aligned} \tag{5.21}$$

So (5.19), (5.20) and (5.21) imply that

Proposition 5.4

$$\frac{d^2}{dt^2}\mathcal{G}(\Sigma_t)|_{t=0} = 2C^2\int_\Sigma (\partial_t|\mathring{P}_r|)^2|_{t=0} - 2\int_\Sigma [\partial_t(s_r - \overline{s_r})]^2|_{t=0}. \tag{5.22}$$

By (2.5), (5.1), (5.13) and (5.15),

$$\begin{aligned}
\frac{\partial}{\partial t}(s_r)|_{t=0} &= -\lambda^{r-1}\binom{n-1}{r-1}\mathrm{trHess} f|_{t=0} - f\lambda^{r-1}\binom{n-1}{r-1}\lambda^2 n - ncf\lambda^{r-1}\binom{n-1}{r-1}\\
&= -\binom{n-1}{r-1}\lambda^{r-1}(\Delta f + n\lambda^2 f + ncf).
\end{aligned} \tag{5.23}$$

Note on Σ, $A = \lambda I$. Hence on Σ, $A(\mathrm{Hess} f) = (\mathrm{Hess} f)A$. This property let us prove the following conclusion.

Proposition 5.5

$$\partial_t(A^m)_i^j|_{t=0} = -m[(Hess f)A^{m-1} + fA^{m+1} + cf(A^{m-1})]_i^j|_{t=0} \tag{5.24}$$

$$= -m\lambda^{m-1}[Hess f + \lambda^2 fI + cfI]_i^j|_{t=0} \tag{5.25}$$

Proof We give the argument by induction. The conclusion holds for $m = 1$. Suppose $\partial_t(A^m)_i^j|_{t=0} = -m[(\mathrm{Hess} f)A^{m-1} + fA^{m+1} + cf(A^{m-1})]_i^j|_{t=0}$. Then

$$\begin{aligned}
\partial_t(A^{m+1})_i^j|_{t=0} &= \partial_t\left[A_i^k(A^m)_k^j\right]|_{t=0}\\
&= (\partial_t A_i^k)(A^m)_k^j|_{t=0} + A_i^k\left[\partial_t(A^m)_k^j\right]|_{t=0}\\
&= -[\mathrm{Hess} f + fA^2 + cfI]_i^k(A^m)_k^j|_{t=0}\\
&\quad -mA_i^k[(\mathrm{Hess} f)A^{m-1} + fA^{m+1} + cf(A^{m-1})]_k^j|_{t=0}\\
&= -(m+1)[(\mathrm{Hess} f)A^m + fA^{m+2} + cf(A^m)]_i^j|_{t=0}.
\end{aligned}$$

By induction, (5.24) holds. Take $A = \lambda I$. Then

$$\partial_t(A^m)_i^j|_{t=0} = -m[\lambda^{m-1}(\mathrm{Hess} f) + f\lambda^{m+1}I + cf\lambda^{m-1}I]_i^j|_{t=0},$$

which is just (5.25). □

Proposition 5.5 implies the following

Proposition 5.6

$$\partial_t(\mathring{P}_r)_i^j|_{t=0} = \lambda^{r-1}\binom{n-2}{r-1}(\mathring{Hess} f)_i^j|_{t=0}, \tag{5.26}$$

$$|\partial_t \mathring{P}_r|^2|_{t=0} = \lambda^{2(r-1)} \binom{n-2}{r-1}^2 |\mathring{H}ess f|^2|_{t=0}, \tag{5.27}$$

where $\mathring{H}ess f = Hess f - \frac{\Delta f}{n} I$.

Proof By the definition (2.4) of P_r,

$$\begin{aligned}
\partial_t (P_r)_i^j|_{t=0} &= \partial_t \left(\sum_{m=0}^{r-1} (-1)^m s_{r-m} (A^m)_i^j \right)|_{t=0} \\
&= \sum_{m=0}^{r-1} (-1)^m (\partial_t s_{r-m})(A^m)_i^j|_{t=0} + \sum_{m=0}^{r} (-1)^m s_{r-m} \partial_t (A^m)_i^j|_{t=0} \\
&= \sum_{m=0}^{r-1} (-1)^m \left[-\binom{n-1}{r-m-1} \lambda^{r-m-1} (\Delta f + nf\lambda^2 + ncf) \lambda^m I \right] \\
&\quad + \sum_{m=0}^{r} (-1)^m \binom{n}{r-m} \lambda^{r-m} [-m\lambda^{m-1} (\mathrm{Hess} f + f\lambda^2 I + cfI)].
\end{aligned} \tag{5.28}$$

By (5.4) and (5.7),

$$\begin{aligned}
&\partial_t (P_r)_i^j|_{t=0} \\
&= -\binom{n-2}{r-1} \lambda^{r-1} (\Delta f + nf\lambda^2 + ncf) I + \lambda^{r-1} \binom{n-2}{r-1} [\mathrm{Hess} f + f\lambda^2 I + fcI] \\
&= -\lambda^{r-1} \binom{n-2}{r-1} [(\Delta f) I + (n-1) f\lambda^2 I + (n-1) cfI - \mathrm{Hess} f]_i^j.
\end{aligned} \tag{5.29}$$

Note $\mathring{P}_r = P_r - \frac{n-r}{n} s_r g$ and $\partial_t g_i^j = 0$. By (5.23) and (5.29), we have

$$\begin{aligned}
\partial_t (\mathring{P}_r)_i^j|_{t=0} &= \left[\partial_t (P_r)_i^j - \frac{n-r}{n} (\partial_t s_r) g_i^j - \frac{n-r}{n} s_r \partial_t g_i^j \right]|_{t=0} \\
&= -\lambda^{r-1} \binom{n-2}{r-1} [(\Delta f) I + (n-1) f\lambda^2 I + (n-1) cfI - \mathrm{Hess} f]_i^j \\
&\quad + \frac{n-r}{n} \binom{n-1}{r-1} \lambda^{r-1} (\Delta f + nf\lambda^2 + ncf) I_i^j \\
&= \lambda^{r-1} \binom{n-2}{r-1} \left[\mathrm{Hess} f - \frac{1}{n} (\Delta f) I + (n-1) cfI \right]_i^j \\
&= \lambda^{r-1} \binom{n-2}{r-1} (\mathring{\mathrm{H}}\mathrm{ess} f)_i^j.
\end{aligned} \tag{5.30}$$

(5.30) yields

$$\begin{aligned}
|\partial_t \mathring{P}_r|^2|_{t=0} &= \sum_{i,j=1}^{n} \partial_t (\mathring{P}_r)_i^j|_{t=0} \partial_t (\mathring{P}_r)_j^i|_{t=0} \\
&= \lambda^{2(r-1)} \binom{n-2}{r-1}^2 |\mathring{\mathrm{H}}\mathrm{ess} f|^2
\end{aligned} \tag{5.31}$$

□

Take $\varphi = s_r$ in Prop 4.5. It holds that

$$\partial_t \overline{s_r}|_{t=0} = [\overline{fH(s_r - \overline{s_r})} + \overline{\partial_t s_r}]|_{t=0} = \overline{\partial_t s_r}|_{t=0}$$
$$= -\lambda^{r-1}\binom{n-1}{r-1}\overline{\Delta f + n\lambda^2 f + ncf}|_{t=0}$$
$$= -\lambda^{r-1}\binom{n-1}{r-1}(n\lambda^2 + c)\overline{f}$$

We will choose f later so that $\int_\Sigma f = 0$. For such f,

$$\partial_t \overline{s_r}|_{t=0} = 0. \tag{5.32}$$

Using (5.23), (5.27) and (5.32), by Proposition 5.22, we may calculate the second variation of $\mathcal{G}$ at $t = 0$ as follows:

$$\begin{aligned}\frac{1}{2}\frac{d^2}{dt^2}\mathcal{G}(\Sigma_t)|_{t=0} &= C^2\int_\Sigma |\partial_t \mathring{\mathrm{P}}_r|^2|_{t=0} - \int_\Sigma [\partial_t(s_r - \overline{s_r})]^2|_{t=0}\\
&= \lambda^{2(r-1)}\left[C^2\binom{n-2}{r-1}^2\int_\Sigma |\mathring{\mathrm{H}}\mathrm{ess}\, f|^2 - \binom{n-1}{r-1}^2\int_\Sigma (\Delta f + nf\lambda^2 + ncf)^2\right]\\
&= \lambda^{2(r-1)}C^2\binom{n-2}{r-1}^2\left[\frac{n-1}{n}\int_\Sigma(\Delta f)^2 + \frac{n-1}{sn_c^2(a)}\int_\Sigma f\Delta f\right]\\
&\quad -\lambda^{2(r-1)}\binom{n-1}{r-1}^2\left[\int_\Sigma(\Delta f)^2 + 2n(\lambda^2+c)\int_\Sigma f\Delta f + n^2(\lambda^2+c)^2\int_\Sigma f^2\right].\end{aligned} \tag{5.33}$$

By (5.6), $(n-1)\binom{n-2}{r-1} = (n-r)\binom{n-1}{r-1}$. So it holds that

$$\begin{aligned}&\frac{1}{2}\frac{d^2}{dt^2}\mathcal{G}(\Sigma_t)|_{t=0}\\
&= \lambda^{2(r-1)}\binom{n-1}{r-1}^2\left[\left(\frac{C^2(n-r)^2}{n(n-1)} - 1\right)\int_\Sigma(\Delta f)^2\right.\\
&\quad \left. + \left(C^2\frac{(n-r)^2}{sn_c^2(a)(n-1)} - 2n(\lambda^2+c)\right)\int_\Sigma f\Delta f - n^2(\lambda^2+c)^2\int_\Sigma f^2\right]\\
&= \lambda^{2(r-1)}\binom{n-1}{r-1}^2\left\{\alpha\int_\Sigma(\Delta f)^2 + \beta\int_\Sigma f\Delta f + \gamma\int_\Sigma f^2\right\}\end{aligned} \tag{5.34}$$

where $\alpha = \frac{C^2(n-r)^2}{n(n-1)} - 1$, $\beta = C^2\frac{(n-r)^2}{sn_c^2(a)(n-1)} - 2n(\lambda^2+c)$, $\gamma = -n^2(\lambda^2+c)^2$.

If $C < \sqrt{\frac{n(n-1)}{(n-r)^2}}$, then $\alpha < 0$. Similar to the proof of Theorem 4.1, we can prove that

Theorem 5.1 *Let M_c^{n+1} be a space form of the constant sectional curvature c. Fix $2 \le r \le n-1$. Given $C < \sqrt{\frac{n(n-1)}{(n-r)^2}}$, there exists a closed totally umbilical hypersurface Σ in M and a deformation $F(\cdot, t)$ of Σ such that*

$$\frac{d^2}{dt^2}\mathcal{G}(\Sigma_t)|_{t=0} < 0,$$

where $\mathcal{G}(\Sigma_t)$ *is given by* (5.3).

The proof of Theorem 1.6 Let $F(x,t)$ be the deformation given in Theorem 5.1 with $\Sigma_0 = \mathbb{S}^n$ and $C = C_1 < \sqrt{\frac{n(n-1)}{(n-r)^2}}$. Then Theorem 5.1 implies that for t sufficiently small,

$$\int_{\Sigma_t} (s_r - \overline{s_r})^2 > C_1^2 \int_{\Sigma_t} |P_r - \frac{(n-r)s_r}{n} g|^2. \tag{5.35}$$

Take $(\Sigma_1)_t = \Sigma_t$ for sufficiently small t. So (1.9) holds.

Given $C_2 < \sqrt{n}$, let $C_1^2 = \frac{n}{(n-r)^2}(C_2^2 - 1)$. Then $C_1 < \sqrt{\frac{n(n-1)}{(n-r)^2}}$. By (5.35) and the identity:

$$|P_r - \frac{(n-r)\overline{s_r}}{n} g|^2 = |P_r - \frac{(n-r)s_r}{n} g|^2 + \frac{(n-r)^2}{n}(s_r - \overline{s_r})^2, \tag{5.36}$$

it holds that there exists a deformation Σ_t, denoted by $(\Sigma_2)_t$ so that

$$\begin{aligned}\int_{\Sigma_t} |P_r - \frac{(n-r)\overline{s_r}}{n} g|^2 &> \left(1 + \frac{(n-r)^2 C_1^2}{n}\right)\int_{\Sigma_t} |P_r - \frac{(n-r)s_r}{n} g|^2 \\ &= C_2^2 \int_{\Sigma_t} |P_r - \frac{(n-r)s_r}{n} g|^2. \end{aligned} \tag{5.37}$$

Clearly, the Ricci curvature Ric_{Σ_t} of Σ_t is positive for t sufficiently small. □

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