ISOPERIMETRIC ESTIMATES IN LOW DIMENSIONAL RIEMANNIAN PRODUCTS

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ABSTRACT. Let $(T^k, h_k) = (S^1_{r_1} \times S^1_{r_2} \times ... \times S^1_{r_k}, dt_1^2 + dt_2^2 + ... + dt_k^2)$ be flat tori, $r_k \geq ... \geq r_2 \geq r_1 > 0$ and $(\mathbb{R}^n, g_E)$ the Euclidean space with the flat metric. We compute the isoperimetric profile of $(T^2 \times \mathbb{R}^n, h_2 + g_E)$, $2 \leq n \leq 5$, for small and big values of the volume. These computations give explicit lower bounds for the isoperimetric profile of $T^2 \times \mathbb{R}^n$. We also note that similar estimates for $(T^k \times \mathbb{R}^n, h_k + g_E)$, $2 \leq k \leq 5$, $2 \leq n \leq 7-k$, may be computed, provided estimates for $(T^{k-1} \times \mathbb{R}^{n+1}, h_{k-1} + g_E)$ exist. We compute this explicitly for $k = 3$. We use symmetrization techniques for product manifolds, based on work of A. Ros ([19]) and F. Morgan ([10]).

1. Introduction

The isoperimetric problem is a classical question in differential geometry. An isoperimetric region of volume t, $0 < t < V^n(M)$, in a manifold (M^n, g), is a closed region Ω of volume $V^n(\Omega) = t$, such that its boundary area is minimal among the compact hypesurfaces $\Sigma \subset M$ enclosing a region of volume t. Throughout the article, the volume of a closed region $\Omega \subset M^n$ will mean $n-$dimensional Riemannian measure of Ω and we will refer to them as $V^n(\Omega)$. On the other hand, the area of a closed region Ω in the manifold M^n will mean the $(n-1)-$dimensional Riemannian measure of $\partial\Omega$ and and we will denote it by $V^{n-1}(\partial\Omega)$.

Given a Riemannian manifold (M^n, g) of volume V, the isoperimetric function or isoperimetric profile of (M, g) is the function $I_{(M,g)} : (0, V) \to (0, \infty)$ given by

$$I_{(M,g)}(t) = \inf\{V^{n-1}(\partial U) : V^n(U) = t, U \subset M^n, U \textit{ a closed region}\}.$$

Note that the isoperimetric profile may be defined for manifolds of infinite volume. We will simply write I_M when the metric g is understood from context. A more detailed treatment of this subject may be found in [19].

Although a classical problem, the isoperimetric profile is known for very few manifolds. It is known explicitly, for example, for space forms $(\mathbb{R}^n, g_E)$, (S^n, g_0), $(\mathbb{H}^n, g_H)$, where g_E, g_0 and g_H are the Euclidean, the round and the hyperbolic metrics, respectively. Other examples include cylinders of the type $(S^n \times \mathbb{R}, g_0 + dt^2)$ by the work of R. Pedrosa [13], and for the Riemannian product of a low dimensional space form with S^1, i.e., $(S^1 \times \mathbb{R}^n, dt^2 + g_E)$, $(S^1 \times S^n, dt^2 + g_0)$, $(S^1 \times \mathbb{H}^n, dt^2 + g_H)$ $(2 \leq n \leq 7)$, by the work of R. Pedrosa and M. Ritoré [14]. Other results in this direction include lower bounds for isoperimetric profiles or characterizations of isoperimetric regions, see for example, [11], [12], [15] and [17]. Nevertheless, for many seemingly simple products like $(S^2 \times \mathbb{R}^2, g_0 + g_E)$ or $(S^1 \times S^1 \times \mathbb{R}^2, dt^2 + ds^2 + g_E)$, the explicit isoperimetric profiles are not known.

Let (T^2, h_2) be a standard flat torus, $T^2 = \mathbb{R}^2/\Gamma$, with Γ the orthogonal lattice generated by $\{(2\pi r_1, 0), (0, 2\pi r_2)\}$, $r_1, r_2 > 0$. For example, the precise isoperimetric profile of $(T^2 \times \mathbb{R}, h_2 + dt^2)$ is not known, and is conjectured to be a profile generated by regions such as spheres (B^3_R), cylinders $(B^2_R \times [0, r])$ and planes $(T^2 \times [0, r])$, $R, r > 0$. This conjecture was proven to be true for small volumes $0 < v < v_1^*$, for some $v_1^* > 0$ by the work of L. Hauswirth, J. Pérez and A. Ros, in [6]. Moreover, one may notice that the conjecture is also true for big volumes $v > v_1^{**}$, for some $v_1^{**} > 0$, through an immediate application of the Ros product Theorem ([19], Theorem 3.7), and a comparison with the isoperimetric profile of $S^2 \times \mathbb{R}$, computed by R. Pedrosa [13]. More precisely, let (T^2, h) be the flat torus with lattice generated by $\{(2\sqrt{\pi}, 0), (0, 2\sqrt{\pi})\}$. By direct computation $V^2(T^2) = 4\pi = V^2(S^2)$ and $I_{S^2} \leq I_{T^2}$. Being S^2 a model metric, we may apply the Ros product Theorem to the inequality. This yields $I_{S^2\times\mathbb{R}} \leq I_{T^2\times\mathbb{R}}$.

Now, let $B^n_R \subset \mathbb{R}^n$ denote a ball of radius $R > 0$, and let $f_n : [0, \infty) \to [0, \infty)$ be the function given by

$$f_n(v) = V^{n+1}(\partial(T^2 \times B^n_R)), \tag{1}$$

with R such that $v = V^{n+2}(T^2 \times B^n_R)$. With this notation, since $T^2 \times B^1_R$ are actual closed regions in $T^2 \times \mathbb{R}$, one has

$$I_{S^2\times\mathbb{R}}(v) \leq I_{T^2\times\mathbb{R}}(v) \leq f_1(v).$$

Explicit computations of $I_{S^2\times\mathbb{R}}$ in the before cited work of R. Pedrosa [13], show that for $v \geq v^{**}$, $I_{S^2\times\mathbb{R}}(v) = f_1(v)$, with $v^{**} \approx 16.66$. It follows that $I_{T^2\times\mathbb{R}}(v) = f_1(v)$ for $v \geq v^{**}$.

We may resume the above discussion in the following Theorem.

Theorem 1.1. (Theorem 18 in [6], together with [13] and [19])
Let (T^2, h) be a standard flat torus, $T^2 = \mathbb{R}^2/\Gamma$, with Γ the orthogonal lattice generated by $\{(2\sqrt{\pi}, 0), (0, 2\sqrt{\pi})\}$. There are some $v_1^*, v_1^{**} > 0$ such that the isoperimetric profile of $(T^2 \times \mathbb{R}, h + dr^2)$ satisfies the following. For $v < v_1^*$, $I_{T^2\times\mathbb{R}}(v) = I_{\mathbb{R}^3}(v)$ and for $v > v_1^{**}$, $I_{T^2\times\mathbb{R}^n}(v) = f_1(v)$. Explicit estimates are $v_1^* = \frac{32\pi^{5/2}}{81} \approx 6.91$ and $v_1^{**} \approx 16.66$.

In this article we paint a similar picture for the Riemannian manifold $(T^2 \times \mathbb{R}^n, h_2 + g_E)$, for $2 \leq n \leq 5$ and g_E the Euclidean metric on $\mathbb{R}^n$. Our first result is the following.

Theorem 1.2. *Let (T^2, h_2) be a standard flat torus, $T^2 = \mathbb{R}^2/\Gamma$, with Γ the orthogonal lattice generated by $\{(0, 2\pi r_1), (2\pi r_2, 0)\}$, $0 < r_1 \leq r_2$. For $2 \leq n \leq 5$, there are some $\tilde{v}_n^*$ and $\tilde{v}_n^{**}$ such that the isoperimetric profile of $(T^2 \times \mathbb{R}^n, h_2 + g_E)$ satisfies the following. For $v \leq \tilde{v}_n^*$, $I_{T^2\times\mathbb{R}^n}(v) = I_{S^1\times\mathbb{R}^{n+1}}(v)$ and for $v \geq \tilde{v}_n^{**}$, $I_{T^2\times\mathbb{R}^n}(v) = f_n(v)$.*

Moreover, our proof gives simple formulas to compute explicit lower bounds for $\tilde{v}_n^*$ and upper bounds for $\tilde{v}_n^{**}$, as functions only of n, r_1, r_2. For example, a lower bound

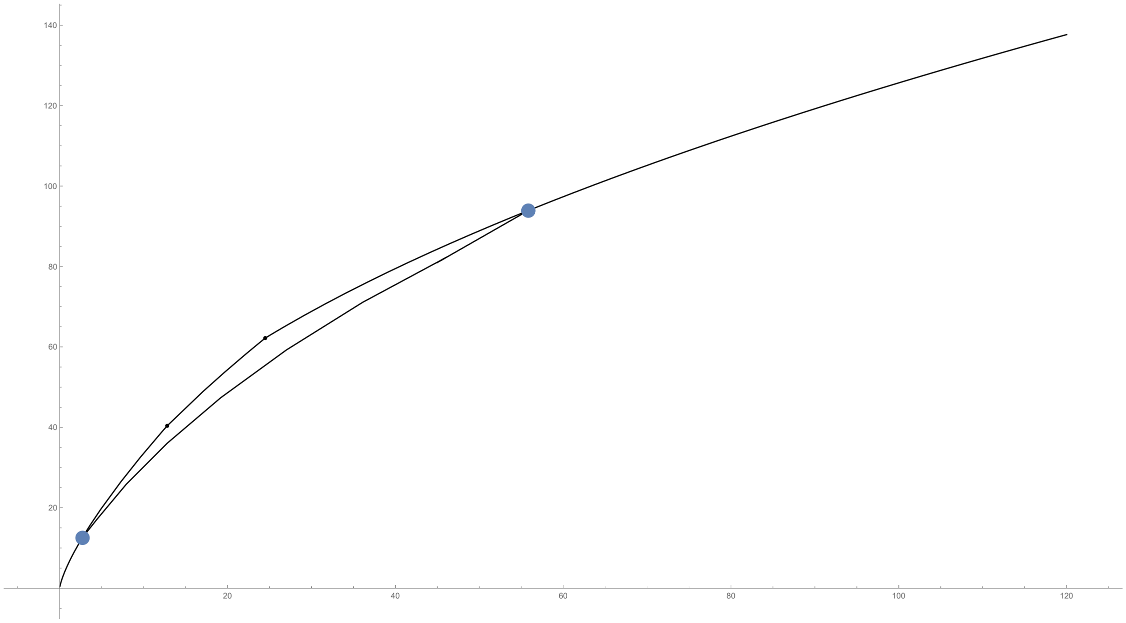

FIGURE 1. Before $v_2^* \approx 2.7$ and after $v_2^{**} \approx 55.8$, the isoperimetric profile of $(T^2 \times \mathbb{R}^2, h + g_E)$ is known precisely. In the interval between v_2^* and v_2^{**}, is bounded above and below. See example 3.9 for details. (T^2, h) is a standard flat torus, $T^2 = \mathbb{R}^2/\Gamma$, with Γ the orthogonal lattice generated by $\{(2\sqrt{\pi}, 0), (0, 2\sqrt{\pi})\}$. g_E is the Euclidean metric.

for $\tilde{v}_n^{**}$ is $v_n^{**} = \max\{a_n, b_n\}$, where a_n is such that $I_{S^1_{r_1} \times \mathbb{R}^{n+1}}(a_n) - f_n(a_n) = 2\beta_n(r_1)$, and b_n such that $I_{S^1_{r_2} \times \mathbb{R}^{n+1}}(b_n) - f_n(b_n) = 2\beta_n(r_2)$. $\beta_n(r)$ being given by eq. (3).

On the other hand, an upper bound for $\tilde{v}_n^*$ is $v_n^* = \min\{c_n, v_s\}$, where $v_s = \min\{V^{n+2}(S^1_{r_1} \times B^{n+1}_{\pi r_2}), V^{n+2}(B^{n+2}_{\pi r_1})\}$ and c_n is such that $I_{S^1_{r_1} \times S^1_{r_2} \times \mathbb{R}^{n+1}}(c_n) = K^*$, where $K^* > 0$ is given by Lemma 3.4. See the proof of Theorem 1.2 for details on these estimates. Numerical estimates for $v_2^* \leq \tilde{v}_2^*$ and $v_2^{**} \geq \tilde{v}_2^*$ for $r_1 = r_2 = 1$ are $v_2^* \approx 5.25$, and $v_2^{**} \approx 70.12$.

The bounds for $\tilde{v}_n^*$ and $\tilde{v}_n^{**}$ we give are not optimal. In fact, we conjecture that $\tilde{v}_n^* = \tilde{v}_n^{**}$. That is, $I_{S^1_{r_1} \times S^1_{r_2} \times \mathbb{R}^n}(v) = I_{scp}(v)$, where $I_{scp}(v) = \min\{I_{S^1_{r_1} \times \mathbb{R}^{n+1}}(v), f_n(v)\}$.

Through the same symmetrization techniques one can obtain corresponding results for $T^3 \times \mathbb{R}^n$, based on the estimates for the isoperimetric profile of $T^2 \times \mathbb{R}^n$. We first define a corresponding function for the area of regions of the type $V^{n+3}(\partial(T^3 \times B^n_R))$. Given a volume v consider the function $g_n(v) = V^{n+2}(\partial(T^3 \times B^n_R))$, with R such that $V^{n+3}(T^3 \times B^n_R) = v$.

Theorem 1.3. *Let (T^3, h_3) be a standard flat k-torus, $T^3 = S^1_{r_1} \times S^1_{r_2} \times S^1_{r_3}$, $r_1 \leq r_2 \leq r_3$. Let $2 \leq n \leq 4$. Suppose that there are some v_n^*, v_n^{**}, with $v_n^{**} \geq v_n^* > 0$, such that the following is satisfied. For $v \leq v_n^*$, $I_{S^1_{r_1} \times S^1_{r_2} \times \mathbb{R}^n}(v) = I_{S^1_{r_1} \times \mathbb{R}^{n+1}}(v)$. For $v \geq v_n^{**}$, $I_{S^1_{r_1} \times S^1_{r_2} \times \mathbb{R}^n}(v) = f_n(v)$.*

Then, there are some $\tilde{u}_n^*, \tilde{u}_n^{**}$, *with* $\tilde{u}_n^{**} \geq \tilde{u}_n^* > 0$, *such that the following is satisfied. For* $v \leq \tilde{u}_n^*$, $I_{S^1_{r_1} \times \mathbb{R}^{n+2}}(v) = I_{S^1_{r_1} \times S^1_{r_2} \times S^1_{r_3} \times \mathbb{R}^n}(v)$. *And for* $v \geq \tilde{u}_n^{**}$, $I_{S^1_{r_1} \times S^1_{r_2} \times S^1_{r_3} \times \mathbb{R}^n}(v) = g_n(v)$.

The proof of Theorem 1.3 gives explicit estimates for lower bounds for $\tilde{u}_n^*$ and upper bounds for $\tilde{u}_n^{**}$, based on those for v_n^* and v_n^{**}. Of course, one may combine the ideas behind the proofs of Theorems 1.2 and 1.3 in an inductive way and obtain corresponding results for $T^k \times \mathbb{R}^n$, $2 \leq k \leq 5$, $2 \leq n \leq 7-k$, based on the estimates for $(T^{k-1} \times \mathbb{R}^n, h + g_E)$.

The fact that, for big volumes, regions of the type $M \times B_R^n$ are isoperimetric regions of manifolds the type $M^k \times \mathbb{R}^n$, where M^k is compact, was known to be true in general (see for example the work of J. Gonzalo [4]). Nevertheless, no explicit estimates of how big the volume should be, in order for this to happen, were known. On the other hand, isoperimetric regions with small volumes were studied in the case $T^2 \times \mathbb{R}$, in [6], using symmetries and properties exclusive of three manifolds. Our approach is different, based on symmetrization techniques like the Ros product Theorem [19] and others introduced by F. Morgan in [10]. We also treat the more general case $T^2 \times \mathbb{R}^n$.

Estimates for v_n^* and v_n^{**} give a good understanding of the general shape of the isoperimetric profile of $(T^2 \times \mathbb{R}^n, h_2 + g_E)$. For example, figure 1 shows lower and upper bounds for the graphic of the isoperimetric profile of $(T^2 \times \mathbb{R}^2, h + g_E)$; based on computations of v_2^* and v_2^{**}.

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2. Notation and background

Existence and regularity of isoperimetric regions is a fundamental result due to the works of Almgren [1], Grüter [5], Gonzalez, Massari, Tamanini [3], (see also Morgan [9], Ros [19]).

Theorem 2.1. *Let* M^n *be a compact Riemannian manifold, or non-compact with* M/G *compact, being* G *the isometry group of* M. *Then, for any* t, $0 < t < V(M)$, *there exists a compact region* $\Omega \subset M$, *whose boundary* $\Sigma = \partial\Omega$ *minimizes area among regions of volume* t. *Moreover, except for a closed singular set of Hausdorff dimension at most* $n-8$, *the boundary* Σ *of any minimizing region is a smooth embedded hypersurface with constant mean curvature.*

Note that $T^2 \times \mathbb{R}^n$ has no boundary, and is compact if it is acted upon by its isometry group. Also since we will only be dealing with the cases $n+2 \leq 7$, every hypersurface Σ enclosing an isoperimetric region will be smooth and of constant mean curvature (CMC). Throughout the article, B_R^n will denote an n-dimensional ball of radius R in $\mathbb{R}^n$.

We will work only with Tori that are Riemannian products: $(T^2, g_2) = (S^1_{r_1} \times S^1_{r_2}, ds^2 + dt^2)$ or $(T^3, g_3) = (S^1_{r_1} \times S^1_{r_2} \times S^1_{r_3}, ds^2 + dt^2 + du^2)$, for some $r_1, r_2, r_3 \in \mathbb{R}^+$. Without loss of generality we will assume $r_1 \leq r_2 \leq r_3$.

The isoperimetric profile of the Riemannian product $S^1_r \times \mathbb{R}^n$, $2 \leq n \leq 7$ is well known, by the work of Pedrosa and Ritoré (Theorem 3.5 in [13]) and is given by

$$I_{S^1_r \times \mathbb{R}^n}(v) = \begin{cases} (1+n)^{\frac{n}{1+n}} \omega_n^{\frac{1}{1+n}} v^{\frac{n}{1+n}}, & \text{if } v \leq \beta_n(r) \\ n^{\frac{n-1}{n}} (2\pi r \omega_{n-1})^{\frac{1}{n}} v^{\frac{n-1}{n}}, & \text{if } v > \beta_n(r) \end{cases} \tag{2}$$

where $\omega_n = V^n(S^n)$ and

$$\beta_n(r) = n^{(n-1)(n+1)} (2\pi r \omega_{n-1})^{n+1} (1+n)^{-n^2} \omega_n^{-n}. \tag{3}$$

Note that $\beta_n(r)$ depends only on n and r. For fixed r and n, $\beta_n(r)$ is the critical number such that for volumes less than $\beta_n(r)$, balls $B^{n+1}_R \subset S^1_r \times \mathbb{R}^n$ are isoperimetric; while for volumes greater than $\beta_n(r)$, regions of the type $S^1_r \times B^n_R \subset S^1 \times \mathbb{R}^n$ are isoperimetric. The isoperimetric profile is continuous. We will denote by $\alpha_n(r)$ the area of the isoperimetric regions of volume $\beta_n(r)$; this is, $\alpha_n(r) = I_{S^1_r \times \mathbb{R}^n}(\beta_n(r))$.

3. The isoperimetric profile of $T^2 \times \mathbb{R}^n$

It was conjectured in [6] that the isoperimetric profile of $T^2 \times \mathbb{R}$ is composed of three parts: for small volumes, the solutions of the isoperimetric problem are spheres (B^3_R), then, for intermediate volumes, cylinders $(S^1 \times B^2_R)$, then, for big volumes, planes $(T^2 \times B^1_R)$. This was called the I_{scp} profile (spheres-cylinders-planes). The conjecture is then that $I_{T^2 \times \mathbb{R}} = I_{scp}$. In the same article, the conjecture was proven to be true for small volumes. The solutions were also proved to be unique and their proof included tori of other types, more general than only orthogonal tori.

We propose a similar conjecture for $T^2 \times \mathbb{R}^n$: for small volumes, spheres B^{n+2}_R are isoperimetric regions, for intermediate volumes, cylinders $(S^1 \times B^{n+1}_R)$, and for big volumes, planes $(T^2 \times B^n_R)$. We will also call this the I_{scp} profile. We will not discuss uniqueness of solutions to the isoperimetric problem. Our results make use of equation (2), so that in the following, n is an integer such that $2 \leq n \leq 7$.

Let Ω be an isoperimetric region in $T^2 \times \mathbb{R}^n = S^1_{r_1} \times S^1_{r_2} \times \mathbb{R}^n$. We may parameterize $S^1_{r_1}$ by $[0, 2\pi r_1]$ and consider the slices Ω_t, $t \in [0, 2\pi\, r_1)$,

$$\Omega_t = \Omega \cap (\{t\} \times S^1_{r_2} \times \mathbb{R}^n).$$

Then for each slice Ω_t, we may compute its $(n+1)$-volume and define a function $F_1 : [0, 2\pi r_1] \to \mathbb{R}$, by $F_1(t) = V^{n+1}(\Omega_t)$ and $F_1(2\pi r_1) = V^{n+1}(\Omega_0)$.

Similarly, one may parameterize $S^1_{r_2}$ by $[0, 2\pi r_2)$ and consider the slices Ω_s, $s \in [0, 2\pi r_2)$:

$$\Omega_s = \Omega \cap (S^1_{r_1} \times \{s\} \times \mathbb{R}^n).$$

Likewise we may define $F_2 : [0, 2\pi\ r_2] \to \mathbb{R}$, by $F_2(s) = V^{n+1}(\Omega_s)$ and $F_2(2\pi r_2) = V^{n+1}(\Omega_0)$. Of course, both F_1 and F_2 are continuous. Let θ_m and θ_M, and σ_m and σ_M, denote the minimum and maximum values of $F_1(t)$ and of $F_2(s)$ respectively.

We start with the following.

Lemma 3.1. *If $\theta_m = 0$ or $\sigma_m = 0$, then*

$$I_{S^1_{r_1}\times\mathbb{R}^{n+1}}(V^{n+2}(\Omega)) \le V^{n+1}(\partial\Omega).$$

Proof. Suppose $\theta_m = 0$. Let $t_0 \in [0, 2\pi r_1]$ be such that $F_1(t_0) = \theta_m = 0$. We construct a new closed region $\Omega^* \subset [t_0, t_0 + 2\pi r_1] \times S^1_{r_2} \times \mathbb{R}^n$, in the following way. For $t \in [0, 2\pi r_1)$, let $\Omega^*_{t_0+t} = \Omega_t$. Also, let $\Omega^*_{t_0+2\pi r_1} = \Omega_{t_0}$. That is, we are adding a copy of Ω_{t_0} at $\{t_0 + 2\pi r_1\} \times S^1_{r_2} \times \mathbb{R}^n$. Since, by hypothesis $V^{n+1}(\Omega_{t_0}) = 0$, we then have $V^{n+1}(\partial\Omega) = V^{n+1}(\partial\Omega^*)$ and $V^{n+1}(\Omega) = V^{n+1}(\Omega^*)$. Note also that Ω^* is a closed region in $\mathbb{R} \times S^1_{r_2} \times \mathbb{R}^n$, by continuity of F_1. It follows that

$$(4)\qquad I_{S^1_{r_2}\times\mathbb{R}^{n+1}}(V^{n+2}(\Omega)) = I_{S^1_{r_2}\times\mathbb{R}^{n+1}}(V^{n+2}(\Omega^*)) \le V^{n+1}(\partial\Omega^*) = V^{n+1}(\partial\Omega).$$

Finally, since $r_1 \le r_2$, eqs. (2) and (4) imply

$$I_{S^1_{r_1}\times\mathbb{R}^{n+1}}(V^{n+2}(\Omega)) \le I_{S^1_{r_2}\times\mathbb{R}^{n+1}}(V^{n+2}(\Omega)) \le V^{n+1}(\partial\Omega).$$

The proof of the case $F_2(s_0) = 0$ is very similar, as in this case Ω can also be embedded isometrically in $S^1_{r_1} \times \mathbb{R} \times \mathbb{R}^n$ as a closed region, by adding an $(n+1)$ zero measure set Ω_{s_0}. Hence in this case we also have $I_{S^1_{r_1}\times\mathbb{R}^{n+1}}(V^{n+2}(\Omega)) \le V^{n+1}(\partial\Omega)$. □

Lemma 3.2. *If $\theta_M \le \beta_n(r_2)$ or $\sigma_M \le \beta_n(r_1)$, then $I_{S^1_{r_1}\times\mathbb{R}^{n+1}}(V^{n+2}(\Omega)) \le V^{n+1}(\partial\Omega)$.*

Proof. We will suppose $\theta_M \le \beta_n(r_2)$; the proof of the case $\sigma_M \le \beta_n(r_1)$ is similar. We will also suppose $\theta_m > 0$ and $\sigma_m > 0$ since the case $\theta_m = 0$ or $\sigma_m = 0$ is treated in Lemma 3.1.

The idea is to symmetrize the isoperimetric region Ω as in the proof of the Ros Product Theorem ([19]). We construct a new region $\Omega^* \subset S^1_{r_1} \times S^1_{r_2} \times \mathbb{R}^n$ by replacing each $\Omega_t \subset \{t\} \times S^1_{r_2} \times \mathbb{R}^n$ with an isoperimetric region in $\{t\} \times S^1_{r_2} \times \mathbb{R}^n$. That is, we let $\Omega^*_t = \{t\} \times B^{n+1}_{R(t)}$, where $R(t) > 0$ is such that $V^{n+1}(\{t\} \times B^{n+1}_{R(t)}) = V^{n+1}(\Omega_t)$. Since $\theta_M \le \beta_n(r_2)$, then $V^{n+1}(\Omega_t) \le \beta_n(r_2)$ for each $t \in [0, 2\pi r_1)$ and hence $\{t\} \times B^{n+1}_{R(t)} \subset \{t\} \times S^1_{r_2} \times \mathbb{R}^n$ for each t.

Also, since $F_1(t)$ is continuous, the region Ω^* is closed in $S^1_{r_1} \times S^1_{r_2} \times \mathbb{R}^n$. Note also that by construction $V^{n+2}(\Omega^*) = V^{n+2}(\Omega)$.

Recall from eq. (2) that for $v \le \beta_n(r_2)$, $I_{S^1_{r_2}\times\mathbb{R}^n}(v) = I_{\mathbb{R}^{n+1}}(v)$. Since $\theta_M \le \beta_n(r_2)$, it follows that for each $t \in [0, 2\pi r_1)$:

$$V^{n+1}(\partial\Omega^*_t) = V^{n+1}(\partial B^{n+1}_{R(t)}) = I_{\mathbb{R}^{n+1}}(V^{n+2}(\Omega_t)) = I_{S^1_{r_2}\times\mathbb{R}^n}(V^{n+2}(\Omega_t)) \le V^{n+1}(\partial\Omega_t).$$

Arguing as in the proof of the Ros Product Theorem, from the last inequality we get

$$V^{n+1}(\partial\Omega^*) \le V^{n+1}(\partial\Omega).$$

Moreover, since $V^{n+2}(\Omega^*) = V^{n+2}(\Omega)$ and Ω^* is a closed region in $S^1_{r_2} \times \mathbb{R}^{n+1}$, we have

$$I_{S^1_{r_2}\times\mathbb{R}^{n+1}}(V^{n+2}(\Omega)) = I_{S^1_{r_2}\times\mathbb{R}^{n+1}}(V^{n+2}(\Omega^*)) \leq V^{n+1}(\partial\Omega^*) \leq V^{n+1}(\partial\Omega).$$

Since $r_1 \leq r_2$ the conclusion of the lemma follows.

□

Recall the definition of $f_n(v)$ by equation (1). We prove the following.

Lemma 3.3. *If $\theta_m \geq \beta_n(r_2)$ or $\sigma_m \geq \beta_n(r_1)$, then*

$$V^{n+1}(\partial\Omega) = f_n(V^{n+2}(\Omega)).$$

Proof. We will prove the case $\theta_m \geq \beta_n(r_2)$. The other one is similar.

Recall from eq. (2) that for $v \geq \beta_n(r_2)$, isoperimetric regions in $S^1_{r_2} \times \mathbb{R}^n$ are of the type $S^1_{r_2} \times B^n_R$. This means $I_{S^1_{r_2}\times\mathbb{R}^n}(v) = V^1(S^1_{r_2})V^{n-1}(\partial B^n_R)$, for some $R > 0$ such that $v = V^1(S^1_{r_2})V^n(B^n_R)$. We symmetrize the isoperimetric region Ω as in the proof of the Ros Product Theorem: we replace each Ω_t in $\{t\} \times S^1_{r_2} \times \mathbb{R}^n$ by a product of $S^1_{r_2}$ and ball $B^n_{R(t)} \subset \mathbb{R}^n$ such that $V^1(S^1_{r_2})V^n(B^n_{R(t)}) = V^{n+1}(\Omega_t)$. We denote the new region in $S^1_{r_1} \times S^1_{r_2} \times \mathbb{R}^n$ by Ω^*. Since $F_1(t)$ is continuous, and for each t we are using a region of the type $S^1_{r_2} \times B^n_{R(t)}$, the region Ω^* is closed in $S^1_{r_1} \times S^1_{r_2} \times \mathbb{R}^n$. Note also that $V^{n+2}(\Omega^*) = V^{n+2}(\Omega)$. And, since for each $t \in [0, 2\pi r_1)$ we have

$$V^n(\partial\Omega^*_t) = I_{S^1_{r_2}\times\mathbb{R}^n}(V^{n+1}(\Omega_t)) \leq V^n(\partial\Omega_t),$$

it follows from the Ros product Theorem that

$$V^{n+1}(\partial\Omega^*) \leq V^n(\partial\Omega). \tag{5}$$

We now symmetrize $\Omega^* \subset S^1_{r_1} \times S^1_{r_2} \times \mathbb{R}^n$ with respect to the other factor, $S^1_{r_2}$. We parameterize $S^1_{r_2}$ by $[0, 2\pi\ r_2)$ and consider the slices Ω^*_s, $s \in [0, 2\pi r_2)$:

$$\Omega^*_s = \Omega^* \cap (S^1_{r_1} \times \{s\} \times \mathbb{R}^n).$$

For each slice we may compute its $(n+1)$-volume and define a function $G : [0, 2\pi r_2] \to \mathbb{R}$, by $G(s) = V^{n+1}(\Omega^*_s)$ for $[0, 2\pi r_2)$ and $G(2\pi r_2) = V^{n+1}(\Omega^*_0)$. Note that G is continuous. Moreover, by construction, for each t and any $s_1, s_2 \in [0, 2\pi r_2]$,

$$\{t\} \times \{s_1\} \times B^n_{R(t)} = \{t\} \times \{s_2\} \times B^n_{R(t)}.$$

This implies that both slices $\Omega^*_{s_1}$ and $\Omega^*_{s_2}$ have the same volume. It follows that $G(s)$ is constant.

We now claim that $G(s) \geq \beta_n(r_2)$: by hypothesis $\theta_m \geq \beta_n(r_2)$, which implies

$$V^{n+2}(\Omega) \geq V^1(S_{r_1})\beta_n(r_2). \tag{6}$$

If the claim were not true, then $G(s) < \beta_n(r_2)$ and we would have

$$V^{n+2}(\Omega^*) = V^1(S_{r_1})G(s) < V^1(S_{r_1})\beta_n(r_2),$$

which is ruled out by inequality (6), since $V^{n+2}(\Omega^*) = V^{n+2}(\Omega)$.

We now construct a new region $\Omega^{**} \subset S^1_{r_1} \times S^1_{r_2} \times \mathbb{R}^n$ by letting each slice $\Omega^{**}_s = S^1_{r_1} \times \{s\} \times B^n_{R_0}$, where R_0 is such that $V^1(S^1_{r_1})V^n(B^n_{R_0}) = G(s)$.

Since $G(s)$ is constant, then R_0 is constant, and we get $V^{n+2}(\Omega^{**}) = V^{n+2}(\Omega^*) = V^{n+2}(\Omega)$. Moreover, by continuity of $G(s)$, the region Ω^{**} is closed. And since $G(s) \geq \beta_n(r_2)$, we have

$$V^n(\partial\Omega_s^{**}) = V^n(\partial(S_{r_1}^1 \times \{s\} \times B_{R_0}^n)) \leq V^n(\partial\Omega_s^*).$$

Hence, using the Ros Product Theorem we get $V^{n+1}(\partial\Omega^{**}) \leq V^{n+1}(\partial\Omega^*)$. And together with eq. (5):

$$V^{n+1}(\partial\Omega^{**}) \leq V^{n+1}(\partial\Omega^*) \leq V^{n+1}(\partial\Omega).$$

Finally, by construction, we have that in fact

$$\Omega^{**} = S_{r_1}^1 \times S_{r_2}^1 \times B_{R_0}^n.$$

It follows that

$$f_n(V^{n+2}(\Omega)) = V^{n+1}(\partial(S_{r_1}^1 \times S_{r_2}^1 \times B_{R_0}^n)) = V^{n+1}(\partial\Omega^{**}) \leq V^{n+1}(\partial\Omega).$$

Being Ω isoperimetric, we conclude that $f_n(V^{n+2}(\Omega)) = V^{n+1}(\partial\Omega)$.

□

We now prove that the case $0 < \theta_m < \beta_n(r_2) < \theta_M$ and $0 < \sigma_m < \beta_n(r_1) < \sigma_M$ cannot occur for small areas of Ω.

Lemma 3.4. *Suppose that $0 < \theta_m < \beta_n(r_2) < \theta_M$ and $0 < \sigma_m < \beta_n(r_1) < \sigma_M$ occurs.*
Then there is some $K^* > 0$ such that $V^{n+1}(\partial\Omega) \geq K^*$. Moreover, K^* is independent of Ω and depends only on r_1, r_2, n. In particular, is given by

$$K^* = \max\{V^1(S_{r_1}^1)\ I_{\mathbb{R}^{n+1}}(\theta^*), V^1(S_{r_2}^1)\ I_{\mathbb{R}^{n+1}}(\sigma^*)\}, \tag{7}$$

where $\theta^* \in (0, \beta_n(r_1))$ is such that

$$\frac{1}{2}V(S_{r_2}^1)I_{\mathbb{R}^{n+1}}(\theta^*) + \theta^* = \beta_n(r_1), \tag{8}$$

and $\sigma^* \in (0, \beta_n(r_2))$ such that

$$\frac{1}{2}V(S_{r_1}^1)I_{\mathbb{R}^{n+1}}(\sigma^*) + \sigma^* = \beta_n(r_2). \tag{9}$$

Remark 3.5. *Since r_1 and r_2 are fixed, equations (8) and (9) are algebraic equations of the type $a\ x^{\frac{n+1}{n+2}} + x = b$, with $a, b, n > 0$. It is straightforward to check that a solution exists and is unique for each equation.*

Proof. We follow a construction by F. Morgan [10], which estimates lower bounds for isoperimetric profiles of products. We consider the product of $(S_{r_1}^1, dt^2)$ with $(S_{r_2}^1 \times \mathbb{R}^n, ds^2 + g_E)$. We start by defining a product manifold $(0, V_1) \times (0, \infty) \subset \mathbb{R}^2$, where $V_1 = V^1(S_{r_1}^1)$. And we equip this 2-dimensional manifold with a model metric in the sense of the Ros product Theorem ([19]). $(0, V_1)$ and $(0, \infty)$ will have Euclidean Lebesgue Measure and Riemannian metric $\frac{1}{2}ds$ and $(\frac{1}{h(x)})dr$ respectively, where $h(x) = I_{S_{r_2}^1 \times \mathbb{R}^{n+1}}(x)$.

To show that this is in fact a model metric, it suffices to prove that in each interval, $(0, V_1)$ and $(0, \infty)$, intervals of the type $(0, t)$, $t > 0$, minimize perimeter, among closed sets S of given Euclidean length t. For the interval $(0,\infty)$ this holds because $h(x)$ is nondecreasing. On the other hand, for the interval $(0,V_1)$, we argue as follows. Suppose $S\subset(0,V_1)$ is a closed set of perimeter t, that is not of the type $(0,t)$; then it must be a locally finite collection of closed intervals; then an interior interval must be at least borderline unstable, because the factor $\frac{1}{2}$ is constant. We conclude that S does not minimize perimeter.

Minkowski content on $(0,V_1)$ and $(0,\infty)$ counts boundary points of intervals with density 2 and $h(x)$, respectively. Similarly, Minkowski content on $(0,V_1)\times(0,\infty)$ has perimeter measured by

$$ds^2 = h^2(v_2)dv_1^2 + 2^2 dv_2^2. \tag{10}$$

It follows from the proof of the Ros Product Theorem that, for any $v>0$, $I_{S^1_{r_1}\times(S^1_{r_2}\times\mathbb{R}^n)}(v)$ is bounded from below by the perimeter $P(E)$ of the boundary δE of some region $E\subset(0,V_1)\times(0,\infty)$. The area of E, $A(E)$, satisfies $v=A(E)$ and δE is a connected boundary curve along which v_2 is nonincreasing and v_1 is nondecreasing. The enclosed region E is on the lower left of δE. Hence

$$P(E)\le I_{S^1_{r_1}\times(S^1_{r_2}\times\mathbb{R}^n)}(v)$$

where

$$P(E)=\int_{\delta E}\sqrt{h^2(v_2)dv_1^2+2^2dv_2^2}. \tag{11}$$

and the area of the region E is given by

$$A(E)=\int\int_E dv_1\,dv_2.$$

Since each term in the square root of eq. (11) is non-negative, we have

$$P(E)\ge 2\int_{\delta E} dv_2 \tag{12}$$

and

$$P(E)\ge\int_{\delta E} h(v_2)dv_1. \tag{13}$$

Now, using the hypothesis $0<\theta_m<\beta_n(r_2)<\theta_M$, we have from eq. (12)

$$P(E)\ge 2(\beta_n(r_2)-\theta_m),$$

and from eq. (13),

$$P(E)\ge\min_{v_2}\{h(v_2)\}\int_{\delta E} dv_1\ge I_{S^1\times\mathbb{R}^n}(\theta_m)\;\; V_1.$$

That is,

$P(E) \geq \max\{2(\beta_n(r_2) - \theta_m), (V_1)\ I_{S^1_{r_2} \times \mathbb{R}^n}(\theta_m)\} = \max\{2(\beta_n(r_2) - \theta_m), (V_1)\ I_{\mathbb{R}^{n+1}}(\theta_m)\}$

where the last equality follows from the fact that $I_{S^1_{r_2} \times \mathbb{R}^n}(\theta_m) = I_{\mathbb{R}^{n+1}}(\theta_m)$ (since $\theta_m < \beta_n(r_2)$). Hence, for the isoperimetric region Ω we have,

$$\max\{2(\beta_n(r_2) - \theta_m), V_1\ I_{\mathbb{R}^{n+1}}(\theta_m)\} \leq P(E) \leq V^{n+1}(\partial\Omega).$$

By Remark 3.5, there is a unique $\theta^* \in (0, \beta_n(r_2))$ such that satisfies $2(\beta_n(r_2) - \theta^*) = V_1 I_{\mathbb{R}^{n+1}}(\theta^*)$, which is eq. (8). Note also that, as functions of θ_m, $2(\beta_n(r_2) - \theta_m)$ is decreasing while $V_1\ I_{\mathbb{R}^{n+1}}(\theta_m)$ is increasing. This yields

$$V_1\ I_{\mathbb{R}^{n+1}}(\theta^*) \leq \max\{2(\beta_n(r_2) - \theta_m), V_1\ I_{\mathbb{R}^{n+1}}(\theta_m)\} \leq P(E) \leq V^{n+1}(\partial\Omega), \tag{14}$$

regardless of the value of θ_m. One may obtain a similar result,

$$V^1(S^1_{r_2})\ I_{\mathbb{R}^{n+1}}(\sigma^*) \leq P(E) \leq V^{n+1}(\partial\Omega), \tag{15}$$

being $\sigma^* \in (0, \beta_n(r_1))$, such that satisfies eq. (9), by following the same analysis for the product of $(S^1_{r_2}, dt^2)$ with $(S^1_{r_1} \times \mathbb{R}^n, ds^2 + g_E)$ and using the hypothesis $0 < \sigma_m < \beta_n(r_1) < \sigma_M$.

Since both eqs. (14) and (15) occur, the conclusion of the Lemma follows.

□

We now prove some lower bounds for $V^{n+1}(\partial\Omega)$ for the case where $0 < \theta_m < \beta_n(r_2) < \theta_M$ and $0 < \sigma_m < \beta_n(r_1) < \sigma_M$ occur.

Lemma 3.6. *Suppose* $0 < \theta_m < \beta_n(r_2) < \theta_M$, *and* $0 < \sigma_m < \beta_n(r_1) < \sigma_M$. *Then*

$$I_{S^1_{r_2} \times \mathbb{R}^{n+1}}(v) - 2\beta_n(r_2) \leq I_{S^1_{r_1} \times S^1_{r_2} \times \mathbb{R}^n}(v) \tag{16}$$

and

$$I_{S^1_{r_1} \times \mathbb{R}^{n+1}}(v) - 2\beta_n(r_1) \leq I_{S^1_{r_1} \times S^1_{r_2} \times \mathbb{R}^n}(v) \tag{17}$$

Proof. We construct a new closed region $\Omega^* \subset [t_m, t_{m+2\pi r_1}] \times S^1_{r_2} \times \mathbb{R}^n \subset \mathbb{R} \times S^1_{r_2} \times \mathbb{R}^n$, with t_m such that $F_1(t_m) = \theta_m > 0$, in the following way. For $t \in (0, 2\pi r_1)$ let $\Omega^*_{t_m+t} = \Omega_t$. Also, let $\Omega^*_{t_m+2\pi r_1} = \Omega_{t_m}$ and $\Omega^*_{t_m} = \Omega_{t_m}$.

Note that $V^{n+2}(\Omega^*) = V^{n+2}(\Omega)$ and $V^{n+1}(\partial\Omega^*) = V^{n+1}(\partial\Omega) + 2V^{n+1}(\Omega_{t_m}) = V^{n+1}(\partial\Omega) + 2\theta_m$.

Let $v = V^{n+2}(\Omega)$. Since Ω^* is actually a closed set in $[t_m, t_{m+2\pi r_1}] \times S^1_{r_2} \times \mathbb{R}^n \subset \mathbb{R} \times S^1_{r_2} \times \mathbb{R}^n$, it follows that

$$I_{S^1_{r_2} \times \mathbb{R}^{n+1}}(v) \leq V^{n+1}(\partial\Omega^*) = V^{n+1}(\partial\Omega) + 2\theta_m$$

Since $\theta_m < \beta_n(r_2)$, and Ω is isoperimetric, we have

$$I_{S^1_{r_2} \times \mathbb{R}^{n+1}}(v) - 2\beta_n(r_2) < I_{S^1_{r_2} \times \mathbb{R}^{n+1}}(v) - 2\theta_m \leq V^{n+1}(\partial\Omega) = I_{S^1_{r_1} \times S^1_{r_2} \times \mathbb{R}^n}(v).$$

Similarly, using $0 < \sigma_m < \beta_n(r_1) < \sigma_M$, one may embed Ω in $S^1_{r_1} \times \mathbb{R}^{n+1}$ by replacing Ω_{s_m} with 2 copies of Ω_{s_m}, where $F(s_m) = \sigma_m$. Following the same argument as before, one gets

$$I_{S^1_{r_1} \times \mathbb{R}^{n+1}}(v) - 2\beta_n(r_1) \leq V^{n+1}(\partial\Omega) = I_{S^1_{r_1} \times S^1_{r_2} \times \mathbb{R}^n}(v).$$

□

We now prove a straightforward lemma that will be useful.

Lemma 3.7. *Let $a, b > 0$, $n \in \mathbb{N}, n > 1$. Consider the function $\varphi(x) = \ x^{\frac{n}{n+1}} - a\, x^{\frac{n-1}{n}}$. There is a unique $x_0 > 0$ such that $\varphi(x_0) = 0$ and $\varphi(x) > 0$ for $x > x_0$. There is a unique $x_1 > 0$ such that $\varphi(x_1) = b$ and $\varphi(x) > b$ for $x > x_1$. Moreover $x_0 < x_1$.*

Proof. For the first claim, we note that for $x > 0$, $\varphi'(x) = 0$ if and only if

$$x = \left(\frac{(n-1)(n+1)}{n^2}\ a \right)^{n(n+1)}.$$

Note that $\varphi(0) = 0 < b$, and $\varphi(x)$ is decreasing for $x > 0$ until $x_1 = (\frac{(n-1)(n+1)}{n^2}\ a)^{n(n+1)} > 0$ and increasing after that. This implies the first and second claims.

For the third claim it suffices to remark that $\varphi(x)$ is still increasing after x_0 and that $\varphi(x_1) < 0 = \varphi(x_0) < b = \varphi(x_1)$; since $b > 0$. It follows that $x_0 < x_1$.

□

Remark 3.8. *Note that since r_1 and r_2 are fixed, equations (18) and (19) are algebraic equations on v of the type $\ v^{\frac{n}{n+1}} - a\ v^{\frac{n-1}{n}} = b$, where $a, b, n > 0$. By Lemma 3.7 they have unique solutions $v_i^{**} > 0$; and $I_{S^1_{r_i} \times \mathbb{R}^{n+1}}(v) - f_n(v) > 2\beta_n(r_i)$ for $v > v_i^{**}$, for $i = 1, 2$. Also, $I_{S^1_{r_i} \times \mathbb{R}^{n+1}}(v) - f_n(v) = 0$ has a unique solution $v_{0_i} > 0$, and $I_{S^1_{r_i} \times \mathbb{R}^{n+1}}(v) > f_n(v)$ for $v > v_{0_i}$, for $i = 1, 2$. Lemma 3.7 also implies $v_{0_i} < v_i^{**}$.*

We now prove Theorem 1.2.

Proof. Let $\Omega \subset S^1_{r_1} \times S^1_{r_2} \times \mathbb{R}^n$ be an isoperimetric region. Consider the functions F_1, F_2 and the values $\theta_m, \theta_M, \sigma_m, \sigma_M$ as before.

We begin with the case of big volumes. Let $v^{**} = \max\{a_n, b_n\}$, where a_n is such that

$$I_{S^1_{r_1} \times \mathbb{R}^{n+1}}(a_n) - f_n(a_n) = 2\beta_n(r_1), \tag{18}$$

and b_n such that

$$I_{S^1_{r_2} \times \mathbb{R}^{n+1}}(b_n) - f_n(b_n) = 2\beta_n(r_2). \tag{19}$$

By Remark 3.8, for $v > v^{**}$,

$$I_{S^1_{r_i} \times \mathbb{R}^{n+1}}(v) - f_n(v) > 2\beta_n(r_i), \tag{20}$$

for $i = 1, 2$. Hence, Lemma 3.6 excludes the case $0 < \theta_m < \beta_n(r) < \theta_M$ and $0 < \sigma_m < \beta_n(r) < \sigma_M$. Remark 3.8 also states that $v^{**} > v_0$, where $v_0 = \max\{v_{0_1}, v_{0_2}\}$ and v_{0_i}

is the unique v such that $I_{S^1_{r_i}\times\mathbb{R}^{n+1}}(v) = f_n(v)$. This implies that for $v > v^{**} > v_0$

$$I_{S^1_{r_i}\times\mathbb{R}^{n+1}}(v) > f_n(v). \tag{21}$$

Since $I_{S^1_{r_1}\times S^1_{r_2}\times\mathbb{R}^n}(v) \leq f_n(v)$ for all $v \geq 0$, inequality (21) excludes the following cases, if $V^{n+2}(\Omega) > v^{**}$:

(1) $\theta_m = 0$ or $\sigma_m = 0$, by Lemma 3.1.
(2) $\theta_M < \beta_n(r_2)$ or $\theta_M < \beta_n(r_1)$, by Lemma 3.2.

Thus, the only case left if $v = V^{n+2}(\Omega) > v^{**}$ is $\theta_m \geq \beta_n(r_2)$ or $\sigma_m \geq \beta_n(r_1)$, which implies $I_{S^1_{r_1}\times S^1_{r_2}\times\mathbb{R}^n}(v) = f_n(v)$ by Lemma 3.3.

We now treat the case of small volumes.

Let $v_s = \min\{V^{n+2}(S^1_{r_1}\times B^{n+1}_{\pi r_2}), V^{n+2}(B^{n+2}_{\pi r_1})\}$.

Isoperimetric regions in $S^1_{r_1}\times\mathbb{R}^{n+1}$ are either regions of the type B^{n+2}_R or $S^1_{r_1}\times B^{n+1}_R$, which are realizable in $S^1_{r_1}\times S^1_{r_2}\times\mathbb{R}^{n+1}$ if $R < \pi r_1 \leq \pi r_2$. This implies $I_{S^1_{r_1}\times S^1_{r_2}\times\mathbb{R}^{n+1}}(v) \leq I_{S^1_{r_1}\times\mathbb{R}^{n+1}}(v)$ for $v \leq v_s$.

Note that for $R < \pi r_2$, $S^1_{r_1}\times B^{n+1}_R$ is a closed region in $S^1_{r_1}\times S^1_{r_2}\times\mathbb{R}^n$.

Hence for $v < v_s$

$$I_{S^1_{r_1}\times S^1_{r_2}\times\mathbb{R}^n}(v) \leq I_{S^1_{r_1}\times\mathbb{R}^{n+1}}(v).$$

Lemma 3.1 implies that if $\theta_m = 0$ or $\sigma_m = 0$, then for $v < v_s$,

$$I_{S^1_{r_1}\times\mathbb{R}^{n+1}}(v) \leq I_{S^1_{r_1}\times S^1_{r_2}\times\mathbb{R}^n}(v) \leq I_{S^1_{r_1}\times\mathbb{R}^{n+1}}(v). \tag{22}$$

By Lemma 3.2, for $v < v_s$, these inequalities are also satisfied if $\theta_M \leq \beta_n(r_2)$ or $\sigma_M \leq \beta_n(r_1)$.

Note also that for $v < \min\{v_s, v_{0_1}\}$, Lemma 3.3 excludes the case $\theta_m \geq \beta_n(r_2)$ or $\sigma_m \leq \beta_n(r_1)$. Otherwise we would have

$$I_{S^1_{r_1}\times\mathbb{R}^{n+1}}(v) > f_n(v) = I_{S^1_{r_1}\times S^1_{r_2}\times\mathbb{R}^n}(v) < I_{S^1_{r_1}\times\mathbb{R}^{n+1}}(v).$$

Finally, let c_n be such that

$$I_{S^1_{r_1}\times\mathbb{R}^n}(c_n) = K^*,$$

where K^* is the constant defined in Lemma 3.4.

Let $v^*_n = \min\{v_s, c_n, v_{0_1}\}$. Then, for $v < v^*_n$,

$$I_{S^1_{r_1}\times S^1_{r_2}\times\mathbb{R}^n}(v) \leq I_{S^1_{r_1}\times\mathbb{R}^{n+1}}(v) < K^*.$$

By Lemma 3.4, this implies that the case $0 \leq \theta_m \leq \beta_n(r_2) \leq \theta_M$ and $0 \leq \sigma_m \leq \beta_n(r_1) \leq \sigma_M$ is excluded for $v < v^*_n$.

We conclude that for $v < v^*_n$,

$$I_{S^1_{r_1}\times S^1_{r_2}\times\mathbb{R}^n}(v) = I_{S^1_{r_1}\times\mathbb{R}^{n+1}}(v).$$

□

We now use these results to compute explicit lower bounds for the isoperimetric profile of a manifold of the type $(T^2\times\mathbb{R}^2, h_2 + g_2)$.

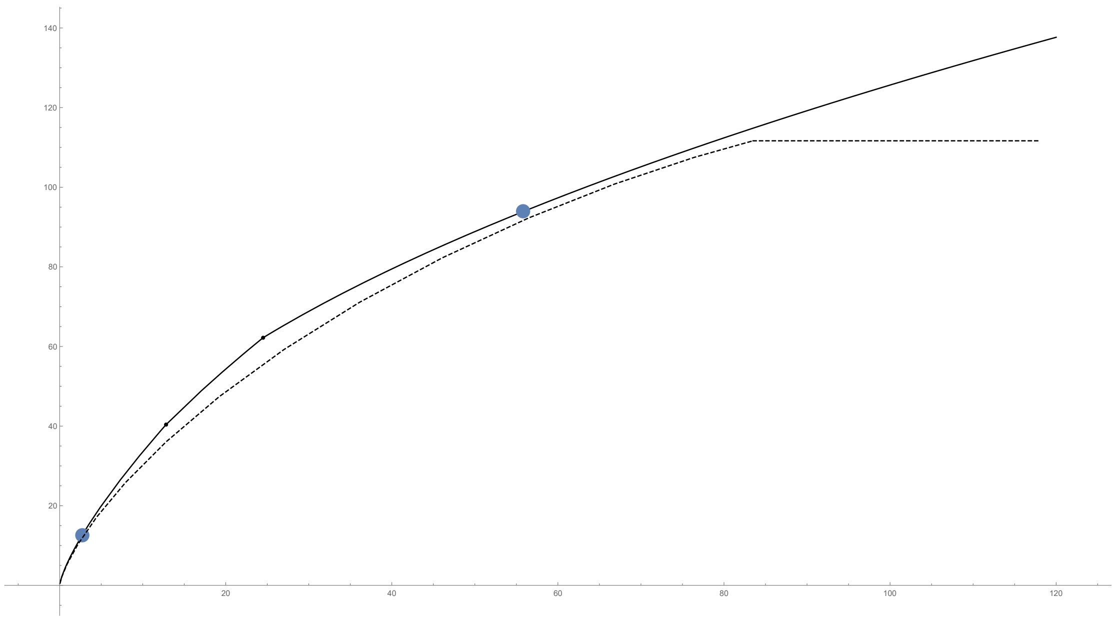

FIGURE 2. $I_{scp}(v)$ profile (solid line) is an upper bound for $I_{T^2\times\mathbb{R}^2}(v)$. The profile $I_{S^3_3\times\mathbb{R}}(v)$ (dashed line) is a lower bound for $I_{T^2\times\mathbb{R}^2}(v)$. Before $v_2^*\approx 2.7$ and after $v^{**}\approx 55.8$, the isoperimetric profile of $(T^2\times\mathbb{R}^2)$ is equal to the $I_{scp}(v)$ profile.

Example 3.9. *Let $(\mathbb{R}^2, g_E)$ be the 2-dimensional Euclidean space and (T^2, h) be the 2-Torus with a flat metric, isometric to $\mathbb{R}/\Gamma$ with Γ the lattice generated by $\{\{2\sqrt{\pi}, 0\}, \{0, 2\sqrt{\pi}\}\}$. Using Theorem 1.2 and its proof, one may make explicit estimates of the isoperimetric profile of $(T^2\times\mathbb{R}^2, h+g_2)$.*

We are in the case $r_1 = r_2 = \frac{1}{\sqrt{\pi}}$, $n = 2$ of Theorem 1.2. By solving eqs. (18) and (19) we get $v_2^{**}\approx 55.84$. We also compute $v_2^*\approx 2.70$, using equations (7), (8) and (9). The $I_{scp}(v)$ profile, given by $I_{scp}(v) = \min\{I_{S^1_{r_1}\times\mathbb{R}^3}(v), f_2(v)\}$, is an upper bound for $I_{T^2\times\mathbb{R}^n}(v)$, moreover, if $v\le v_2^*$ or $v\ge v_2^{**}$, then $I_{T^2\times\mathbb{R}^2}(v) = I_{scp}(v)$. The solid line graphic of figure 2 is the graphic of $I_{scp}(v)$. In this case $f_2(v) = 4\pi\sqrt{v}$.

One may compute lower bounds for the volumes between v_2^* and v_2^{**}. First, since the Ricci curvature of $(T^2\times\mathbb{R}^2, h+g_E)$ is non-negative, it follows from a result by V. Bayle ([2], p. 52) that the isoperimetric profile is concave. This implies that a line joining the points $(v^*, I_{scp}(v^*))$ and $(v^{**}, I_{scp}(v^{**}))$ is also a lower bound for $I_{T^2\times\mathbb{R}^2}(v)$.

A better lower bound for $I_{T^2\times\mathbb{R}^2}(v)$ may be computed in the following way. Since the isoperimetric profiles of (S^2, g_2) and (T^2, h) are known explicitly, it is straightforward to check $I_{S^2}\le I_{T^2}$. Here, g_2 is the round metric with radius $r = 1$. Since S^2 is a model metric, it follows from the Ros symmetrization Theorem [19], that $I_{S^2\times\mathbb{R}^2}\le I_{T^2\times\mathbb{R}^2}$. On the other hand, it was proved in section 2.1 of [18] that $I_{S^3_3\times\mathbb{R}}\le I_{S^2\times\mathbb{R}^2}$, where (S^3_3, g_3) is the 3-sphere with the round metric and radius $r = 3$. It follows that $I_{S^3_3\times\mathbb{R}}\le I_{T^2\times\mathbb{R}^2}$. The isoperimetric profile of $I_{S^3_3\times\mathbb{R}}$ was computed in [13] and its graphic corresponds to the the dashed graphic of figure 2. Moreover, using that the isoperimetric profile of $(T^2 \times \mathbb{R}^2, h + g_E)$ is concave, it follows that any line joining the point $(v^{**}, I_{scp}(v^{**}))$ and the graphic of $I_{S^3_3 \times \mathbb{R}}(v)$ is also a lower bound for $I_{T^2\times\mathbb{R}^2}(v)$. Similarly, any line joining the point $(v^*, I_{scp}(v^*))$ and the graphic of $I_{S^3_3\times\mathbb{R}}(v)$ is also a lower bound for $I_{T^2\times\mathbb{R}^2}(v)$.

These lower bounds gives us a fair idea of the shape of $I_{T^2\times\mathbb{R}^2}$ in the interval (v_2^*, v_2^{**}) and are illustrated in figure 1.

4. The isoperimetric profile of $T^k \times \mathbb{R}^n$

One may follow the arguments of the last section in order to understand the isoperimetric profile of $T^k \times \mathbb{R}^n$ for small and big volumes. In this section we present the proof of Theorem 1.3, that is, the case $k = 3$. The more general case, $2 \le k \le 5$, $2 \le n \le 7 - k$, is similar.

Let $(T^3, h_3) = (S^1_{r_1} \times S^1_{r_2} \times S^1_{r_3}, ds_1^2 + ds_2^2 + ds_3^3)$. Let Ω be an isoperimetric region in $(T^3 \times \mathbb{R}^n, h_3 + g_E)$. We parameterize $S^1_{r_3}$, by $[0, 2\pi r_3)$ and consider slices Ω_t, $t \in [0, 2\pi r_3)$:

$$\Omega_t = \Omega \cap (S^1_{r_1} \times S^1_{r_2} \times \{t\} \times \mathbb{R}^n).$$

Then for each slice Ω_t, we may compute its $(n+2)$-volume and define a function $F : [0, 2\pi\ r_3] \to \mathbb{R}$, by $F(t) = V^{n+2}(\Omega_t)$ and $F(2\pi r_3) = V^{n+2}(\Omega_0)$.

Note that F is continuous. Let η_m and η_M denote the minimum and maximum values of F, respectively.

Lemma 4.1. *If $\eta_m = 0$, then $I_{S^1_{r_1}\times S^1_{r_2}\times\mathbb{R}^{n+1}}(V^{n+3}(\Omega)) \le V^{n+2}(\partial\Omega)$.*

Proof. Suppose $\eta_m = 0$. Let $t_0 \in [0, 2\pi r_3]$ be such that $F(t_0) = \eta_m = 0$. We construct a new closed region $\Omega^* \subset S^1_{r_1} \times S^1_{r_2} \times [t_0, t_0 + 2\pi r_3] \times \mathbb{R}^n$. We denote $\Omega^* \cap (S^1_{r_1} \times S^1_{r_2} \times \{t\} \times \mathbb{R}^n)$ by Ω^*_t. For $t \in [0, 2\pi r_3)$, let $\Omega^*_{t_0+t} = \Omega_t$. Also, let $\Omega^*_{t_0+2\pi r_3} = \Omega_{t_0}$. Ω^* is a closed region by continuity of F. Also, since $V^{n+2}(\Omega_{t_0}) = 0$ we have $V^{n+2}(\partial\Omega) = V^{n+2}(\partial\Omega^*)$ and $V^{n+2}(\Omega) = V^{n+2}(\Omega^*)$. Hence

$$I_{S^1_{r_1}\times S^1_{r_2}\times\mathbb{R}^{n+1}}(V^{n+3}(\Omega)) = I_{S^1_{r_1}\times S^1_{r_2}\times\mathbb{R}^{n+1}}(V^{n+3}(\Omega^*)) \le V^{n+2}(\partial\Omega^*) = V^{n+2}(\partial\Omega). \tag{23}$$

□

Let $w_n^* = \min\{v_n^*, \beta_{n+1}(r_1)\}$, where v_n^* is as in the hypothesis of Theorem 1.3 and $\beta_{n+1}(r_1)$ as in eq. (2). Hence, for $v < w^*$ we have

$$I_{S^1_{r_1}\times S^1_{r_2}\times\mathbb{R}^n}(v) = I_{S^1_{r_1}\times\mathbb{R}^{n+1}}(v) = I_{\mathbb{R}^{n+2}}(v) \tag{24}$$

Lemma 4.2. *If $\eta_M \le w_n^*$, then $I_{S^1_{r_3}\times\mathbb{R}^{n+2}}(V^{n+3}(\Omega)) \le V^{n+2}(\partial\Omega)$.*

Proof. We symmetrize Ω by constructing a region Ω^* as in the proof of Lemma 3.2. We denote $\Omega^* \cap (S^1_{r_1} \times S^1_{r_2} \times \{t\} \times \mathbb{R}^n)$ by Ω^*_t. Let $\Omega^*_t = \{t\} \times B^{n+2}_{R(t)}$, where $R(t) > 0$ is such that $V^{n+2}(\{t\} \times B^{n+2}_{R(t)}) = V^{n+2}(\Omega_t)$.

Note that

$$V^{n+2}(\Omega^*_t) = V^{n+2}(\Omega_t)$$

and since $\eta_M < w_n^*$, each region $\{t\} \times B^{n+1}_{R(t)}$ is isoperimetric in $\{t\} \times S^1_{r_1} \times S^1_{r_2} \times \mathbb{R}^n$, that is $V^{n+1}(\partial\Omega_t^*) \le V^{n+1}(\partial\Omega_t)$.

Arguing as in the proof of the Ros product Theorem ([19]), we get $V^{n+3}(\Omega^*) = V^{n+3}(\Omega)$ and $V^{n+2}(\partial\Omega^*) \le V^{n+2}(\partial\Omega)$.

This implies

$$I_{S^1_{r_3}\times\mathbb{R}^{n+2}}(V^{n+3}(\Omega)) = I_{S^1_{r_3}\times\mathbb{R}^{n+2}}(V^{n+3}(\Omega^*)) \le V^{n+2}(\partial\Omega^*) \le V^{n+2}(\partial\Omega).$$

□

Let g_n be the function given by $g_n(v) = V^{n+2}(\partial(S^1_{r_1} \times S^1_{r_2} \times S^1_{r_3} \times B^n_R))$, where R is such that $V^{n+3}(S^1_{r_1} \times S^1_{r_2} \times S^1_{r_3} \times B^n_R)) = v$.

Lemma 4.3. *Suppose $\eta_m \ge v_n^{**}$. If $V^{n+3}(\Omega) < \beta_n(r_3)V^1(S^1_{r_1})V^1(S^1_{r_2})$, then*

$$I_{S^1_{r_1}\times S^1_{r_2}\times\mathbb{R}^{n+1}}(V^{n+3}(\Omega)) \le V^{n+2}(\partial\Omega).$$

On the other hand, if $V^{n+3}(\Omega) \ge \beta_n(r_3)V^1(S^1_{r_1})V^1(S^1_{r_2})$, then

$$V^{n+2}(\partial\Omega) = g_n(V^{n+3}(\Omega)). \tag{25}$$

Proof. We construct a new region Ω^*. We denote $\Omega^* \cap (S^1_{r_1} \times S^1_{r_2} \times \{t\} \times \mathbb{R}^n)$ by Ω_t^*. Let $\Omega_t^* = \{t\} \times S^1_{r_1} \times S^1_{r_2} \times B^n_{R(t)}$, with $R(t)$ such that $V^{n+2}(S^1_{r_1} \times S^1_{r_2} \times B^n_{R(t)}) = V^{n+2}(\Omega_t)$. Since $\eta_m > v_n^{**}$, regions of the type $S^1_{r_1} \times S^1_{r_2} \times B^n_{R(t)}$ are isoperimetric in $S^1_{r_1} \times S^1_{r_2} \times \mathbb{R}^n$. Arguing as in the proof of the Ros product Theorem ([19]), we get $V^{n+3}(\Omega^*) = V^{n+3}(\Omega)$ and $V^{n+2}(\partial\Omega^*) \le V^{n+2}(\partial\Omega)$.

We now symmetrize Ω^*. Let $(\Omega^*)^p = \Omega^* \cap (\{p\} \times S^1_{r_3} \times \mathbb{R}^n)$ where $p \in S^1_{r_1} \times S^1_{r_2}$.

Let $p, q \in S^1_{r_1} \times S^1_{r_2}$. Note that $(\Omega^*)^p = \bigcup_{t\in S^1_{r_3}} \left(\Omega_t^* \cap (\{p\} \times \{t\} \times B^n_{R(t)})\right)$ and $(\Omega^*)^q = \bigcup_{t\in S^1_{r_3}} \left(\Omega_t^* \cap (\{q\} \times \{t\} \times B^n_{R(t)})\right)$. Since $R(t)$ is the same on both slices we get,

$$V^{n+1}((\Omega^*)^p) = V^{n+1}((\Omega^*)^q). \tag{26}$$

Since p, q where arbitrary, this implies

$$V^{n+3}(\Omega) = V^{n+1}((\Omega_p)V^1(S_{r_1})V^1(S_{r_2}). \tag{27}$$

If $V^{n+3}(\Omega) < \beta_n(r_3)V^1(S^1_{r_1})V^1(S^1_{r_2})$, then by eq. (27), $V^{n+1}((\Omega^*)^p) < \beta_n(r_3)$ and hence balls B^{n+1}_R are isoperimetric regions in $S^1_{r_3} \times \mathbb{R}^n$.

We construct a new region Ω^{**} such that $(\Omega^{**})^p = \{p\} \times B^{n+1}_R$, $p \in S^1_{r_1} \times S^1_{r_2}$ with R such that $V^{n+1}(B^{n+1}_R) = V^{n+1}((\Omega^*)^p)$ (note that R is independent of p, by eq. (26)).

Arguing as in the Ros Product Theorem, we then have $V^{n+3}(\Omega^{**}) = V^{n+3}(\Omega^*) = V^{n+3}(\Omega)$ and $V^{n+2}(\partial\Omega^{**}) \le V^{n+2}(\partial\Omega^*) \le V^{n+2}(\partial\Omega)$. This implies the first part of the Lemma:

$$I_{S^1_{r_1}\times S^1_{r_2}\times\mathbb{R}^{n+1}}(V^{n+3}(\Omega)) \le V^{n+2}(\partial\Omega).$$

On the other hand, if $V^{n+3}(\Omega) \ge \beta_n(r_3)V^1(S^1_{r_1})V^1(S^1_{r_1})$ then by eq. (27), $V^{n+1}((\Omega^*)^p) \ge \beta_n(r_3)$ and hence $S^1_{r_3} \times B^n_R$ are isoperimetric regions in $S^1_{r_3} \times \mathbb{R}^n$.

We then construct a new region Ω^{**} such that $(\Omega^{**})^p = \{p\} \times S^1_{r_3} \times B^n_R$, $p \in S^1_{r_1} \times S^1_{r_2}$, with R such that $V^{n+1}(S^1_{r_3} \times B^n_R) = V^{n+1}((\Omega^*)^p)$ (R is independent of p, by eq. (26)).

Arguing as in the Ros Product Theorem, we get $V^{n+3}(\Omega^{**}) = V^{n+3}(\Omega^*) = V^{n+3}(\Omega)$ and $V^{n+2}(\partial\Omega^{**}) \leq V^{n+2}(\partial\Omega^*) \leq V^{n+2}(\partial\Omega)$. This implies $g_n(V^{n+3}(\Omega)) \leq V^{n+2}(\partial\Omega)$.

Being Ω isoperimetric, we conclude that $g_n(V^{n+3}(\Omega)) = V^{n+2}(\partial\Omega)$.

□

We now prove that the case $\eta_M > w^*_n$ cannot occur for small areas of Ω.

Lemma 4.4. *Suppose that* $\eta_M > w^*_n$.

Then there is some $C^* > 0$ *such that*

$$V^{n+2}(\partial\Omega) > C^*.$$

C^* *is independent of* Ω. *In fact, it depends only on* r_1, r_2, r_3, n *and is given by*

$$C^* = 2(w^*_n - \eta^*)$$

where $\eta^* > 0$ *satisfies*

$$\frac{1}{2}V(S^1_{r_3})I_{\mathbb{R}^{n+2}}(\eta^*) + \eta^* = w^*_n \tag{28}$$

Proof. The proof is similar to that of lemma 3.4. □

Remark 4.5. *By remark 3.5, a solution to equation (28) exists and is unique.*

We now prove a lower bound for the area of the region Ω, $V^{n+2}(\partial\Omega)$, for the case $\eta_m < v^{**}_n$.

Lemma 4.6. *Suppose* $0 < \eta_m < v^{**}_n$. *Then*

$$I_{S^1_{r_1} \times S^1_{r_2} \times \mathbb{R}^{n+1}}(V^{n+3}(\Omega)) - 2v^{**}_n < V^{n+2}(\partial\Omega) \tag{29}$$

Proof. We construct a new closed region $\Omega^* \subset S^1_{r_1} \times S^1_{r_2} \times [t_m, t_{m+2\pi r_3}] \times \mathbb{R}^n \subset S^1_{r_1} \times S^1_{r_2} \times \mathbb{R} \times \mathbb{R}^n$, with t_m such that $F(t_m) = \eta_m > 0$, in the following way. We denote $\Omega^* \cap (S^1_{r_1} \times S^1_{r_2} \times \{t\} \times \mathbb{R}^n)$ by Ω^*_t. For $t \in (0, 2\pi r_3)$ let $\Omega^*_{t_m+t} = \Omega_t$. Also, let $\Omega^*_{t_m+2\pi r_3} = \Omega_{t_m}$ and $\Omega^*_{t_m} = \Omega_{t_m}$.

Note that $V^{n+3}(\Omega^*) = V^{n+3}(\Omega)$ and $V^{n+2}(\partial\Omega^*) = V^{n+2}(\partial\Omega) + 2V^{n+2}(\Omega_{t_m}) = V^{n+2}(\partial\Omega) + 2\eta_m$. Since Ω^* is actually a closed set in $S^1_{r_1} \times S^1_{r_2} \times \mathbb{R} \times \mathbb{R}^n$, it follows that

$$I_{S^1_{r_1} \times S^1_{r_2} \times \mathbb{R}^{n+1}}(V^{n+3}(\Omega)) \leq V^{n+2}(\partial\Omega^*) = V^{n+2}(\partial\Omega) + 2\eta_m$$

Since $\eta_m < v^{**}_n$, we have eq. (29).

□

We now prove Theorem 1.3.

Proof. Let $\Omega \subset S^1_{r_1} \times S^1_{r_2} \times S^1_{r_3} \times \mathbb{R}^n$ be an isoperimetric region. Consider the functions F, g_n and the values η_m, η_M as before.

We begin with the case of big volumes. By Lemma 3.7 and Remark 3.8, there exists u_n^{**} such that for $v > u_n^{**}$

$$I_{S^1_{r_1}\times S^1_{r_2}\times\mathbb{R}^{n+1}}(v) - 2v_n^{**} > g_n(v).$$

Being Ω isoperimetric, we have $V^{n+1}(\partial\Omega) \leq g_n(v)$. Hence, if $V^{n+3}(\Omega) > u_n^{**}$,

$$I_{S^1_{r_1}\times S^1_{r_2}\times\mathbb{R}^{n+1}}(V^{n+3}(\Omega)) - 2v_n^{**} \geq V^{n+2}(\partial\Omega). \tag{30}$$

Hence, eq. (30) and Lemma 4.6 prevents $\eta_m < v_n^{**}$ from happening if $V^{n+3}(\Omega) > u_n^{**}$. This implies that for $V^{n+3}(\Omega) > u_n^{**}$, $\eta_m \geq v_n^{**}$ and by Lemma 4.3, that $V^{n+2}(\partial\Omega) = g_n(V^{n+3}(\Omega))$.

We now treat the case of small volumes.

Suppose $V^{n+2}(\partial\Omega) < C^*$. By Lemma 4.4, $\eta_m < w_n^*$. This implies, by Lemma 4.1 and 4.2, that

$$\min\{I_{S^1_{r_1}\times S^1_{r_2}\times\mathbb{R}^{n+1}(v)}, I_{S^1_{r_3}\times\mathbb{R}^{n+2}}(v)\} \leq I_{S^1_{r_1}\times S^1_{r_2}\times S^1_{r_3}\times\mathbb{R}^{n}}(v).$$

On the other hand, $I_{S^1_{r_1}\times\mathbb{R}^{n+2}}(v) \leq I_{S^1_{r_3}\times\mathbb{R}^{n+2}}(v)$ since $r_1 \leq r_3$ and $I_{S^1_{r_1}\times\mathbb{R}^{n+2}}(v) \leq I_{S^1_{r_1}\times S^1_{r_2}\times\mathbb{R}^{n+1}}(v)$, if $v \leq v_{n+1}^*$.

Hence, if $V^{n+3}(\Omega) \leq v_{n+1}^*$ and $V^{n+2}(\partial\Omega) < C^*$, we have

$$I_{S^1_{r_1}\times\mathbb{R}^{n+2}}(V^{n+3}(\Omega)) \leq I_{S^1_{r_1}\times S^1_{r_2}\times S^1_{r_3}\times\mathbb{R}^{n}}(V^{n+3}(\Omega)).$$

Note also that if $v < V^1(S^1_{r_1})V^n(B_{r_2})$, then isoperimetric regions in $S^1_{r_1}\times\mathbb{R}^{n+2}$ are realizable in $S^1_{r_1}\times S^1_{r_2}\times S^1_{r_3}\times\mathbb{R}^n$. Hence, in this case,

$$I_{S^1_{r_1}\times\mathbb{R}^{n+2}}(V^{n+3}(\Omega)) \geq I_{S^1_{r_1}\times S^1_{r_2}\times S^1_{r_3}\times\mathbb{R}^{n}}(V^{n+3}(\Omega)).$$

Let $u_0 > 0$ be such that $I_{S^1_{r_1}\times\mathbb{R}^{n+2}}(u_0) = C^*$ and $u_n^* = \min\{u_0, v_{n+1}^*, V^1(S^1_{r_1})\, V^{n+2}(B^{n+2}_{r_2}))\}$

We conclude that if $V^{n+3}(\Omega) \leq u_n^*$,

$$I_{S^1_{r_1}\times\mathbb{R}^{n+2}}(V^{n+3}(\Omega)) = I_{S^1_{r_1}\times S^1_{r_2}\times S^1_{r_3}\times\mathbb{R}^{n}}(V^{n+3}(\Omega)).$$

□

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