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Multivariate Analysis

On the Construction of Families of Absolutely Continuous Copulas with Given Restrictions

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The problem of constructing copulas with a given diagonal section has been studied by Sungur and Yang (1996) and Fredricks and Nelsen (1997a,b, 2002). The results of Sungur and Yang are especially relevant because, among other results, they have proven that an Archimedean copula is characterized by its diagonal section. The results obtained by Fredricks and Nelsen allow one to build a singular copula with a given a diagonal section. In all cases, the resulting copulas are symmetric. In this article, we provide a family of absolutely continuous copulas with a fixed diagonal, which can differ from another absolutely continuous copula almost everywhere with respect to Lebesgue measure. It is important to mention that the asymmetry in the proposed methodology is not an issue.

Keywords Absolutely continuous copulas; Copulas; Diagonal sections.

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1. Introduction

A *bivariate copula* is a function $C : [0, 1]^2 \to [0, 1]$ with the following properties:

1. For every u, v in $[0, 1]$,

$$C(u, 0) = 0 = C(0, v), \tag{1}$$

$$C(u, 1) = u \quad \text{and} \quad C(1, v) = v; \tag{2}$$

2. For every u_1, u_2, v_1, v_2 in $[0, 1]$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$,

$$C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0. \tag{3}$$

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It is well known (see, for example, Schweizer and Sklar, 1983) that for every copula C and every (u, v) in $[0, 1]^2$

$$W(u, v) \leq C(u, v) \leq M(u, v), \tag{4}$$

where $W(u, v) := \max(u + v - 1, 0)$ and $M(u, v) := \min(u, v)$. It is also straightforward to show that W and M are themselves copulas, and they are known as the *Fréchet-Hoeffding lower and upper bounds*, respectively.

A copula is also uniformly continuous on its domain (see, for example, Schweizer and Sklar, 1983). That is, for every u_1, u_2, v_1, v_2 in $[0, 1]$

$$|C(u_2, v_2) - C(u_1, v_1)| \leq |u_2 - u_1| + |v_2 - v_1|. \tag{5}$$

The *diagonal section* of a copula is defined by $\delta_C(u) := C(u, u)$. The above definitions and properties have the following straightforward implications for the diagonal section of a copula:

$$\delta_C(0) = 0, \quad \delta_C(1) = 1; \tag{6}$$

$$0 \leq \delta_C(u_2) - \delta_C(u_1) \leq 2(u_2 - u_1), \quad \text{for all } u_1, u_2 \text{ in } [0, 1] \text{ with } u_1 \leq u_2; \tag{7}$$

$$\max(2u - 1, 0) \leq \delta_C(u) \leq u. \tag{8}$$

From now on, any function $\delta : [0, 1] \to [0, 1]$ satisfying (6), (7), and (8) will be called simply a *diagonal*, while the function δ_C will be referred to as the *diagonal section* of a copula C.

If δ is any diagonal, does there exist a copula C whose diagonal section is δ? This question has been answered affirmatively by Sungur and Yang (1996), for the case of Archimedean copulas. A copula C is said to be *Archimedean* if it satisfies the following associativity equation

$$C(C(u, v), w) = C(u, C(v, w)), \quad \text{for all } u, v, w \in [0, 1], \tag{9}$$

and $\delta_C(u) < u$ whenever $0 < u < 1$. The following theorem is due to Ling (1965).

Theorem 1.1 (Ling, 1965). (i) *Let C be an Archimedean copula. Then there exists a strictly decreasing and convex (hence continuous) function $\varphi : (0, 1) \to (0, \infty)$ with $\varphi(1) = 0$ such that for every $(u, v) \in [0, 1]^2$*

$$C(u, v) = \varphi^{[-1]}(\varphi(u) + \varphi(v)), \tag{10}$$

where $\varphi^{[-1]}$ is the pseudo-inverse of φ given by

$$\varphi^{[-1]}(x) := \begin{cases} \varphi^{-1}(x) & 0 \leq x \leq \varphi(0), \\ 0 & \varphi(0) < x < \infty. \end{cases}$$

(ii) *Conversely, if $\varphi : (0, 1) \to (0, \infty)$ is a strictly decreasing and convex function with $\varphi(1) = 0$, and $\varphi^{[-1]}$ is the pseudo-inverse of φ, then the function C defined on $(0, 1)^2$ by (10) and extended to $[0, 1]^2$ by continuity is an Archimedean copula.*

An immediate consequence from Ling's Theorem is that Archimedean copulas are symmetric. The following theorem by Sungur and Yang (1996) gives an alternative representation of the Archimedean class of copulas in terms of their corresponding diagonal sections.

Theorem 1.2 (Sungur and Yang, 1996). *Let C be an Archimedean copula and $\delta_C(u) = C(u, u)$. Then*

$$C(u, v) = \lim_{n\to\infty} \delta_C^n\big(\delta_C^{-n}(u) + \delta_C^{-n}(v) - 1\big), \tag{11}$$

where

$$\delta_C^n := \delta_C \circ \delta_C \circ \cdots \circ \delta_C \quad (n \text{ times}),$$

and

$$\delta_C^{-n} := \delta_C^{-1} \circ \delta_C^{-1} \circ \cdots \circ \delta_C^{-1} \quad (n \text{ times}).$$

Without constraining to the Archimedean class of copulas, Fredricks and Nelsen (1997b) proved that

$$K(u, v) = \min(u, v, (1/2)[\delta(u) + \delta(v)])$$

is, in fact, a copula whose diagonal section is δ. It is immediate to observe that K is symmetric. They also proved, among several other properties, that K is a *singular copula*, that is $\partial^2 K(u, v)/\partial u \partial v = 0$ almost everywhere with respect to Lebesgue measure in $[0, 1]^2$.

Another type of copulas constructed from a given diagonal was presented by Fredricks and Nelsen (2002) known as *The Bertino family of copulas*: For each diagonal δ define B_δ on $[0, 1]^2$ by

$$B_\delta(u, v) := \min(u, v) - \min_{t\in[u,v]} \hat{\delta}(t),$$

where $\hat{\delta}(t) := t - \delta(t)$ for all t in $[0, 1]$. They proved that B_δ is a symmetric singular copula with diagonal section δ. Moreover, they proved that if C is any copula with diagonal section δ, then $B_\delta \le C$ on $[0, 1]^2$.

The two types of copulas constructed with a given diagonal by Fredricks and Nelsen happen to be symmetric and singular. If δ is any diagonal, does there exist a non Archimedean and absolutely continuous copula C whose diagonal section is δ? As defined by Nelsen (1999), C is an *absolutely continuous copula* if

$$C(u, v) = \int_0^u \int_0^v \frac{\partial^2}{\partial s \partial t} C(s, t)\, dt\, ds. \tag{12}$$

In this case, it is common to call $c(u, v) := \partial^2 C(u, v)/\partial u \partial v$ the *copula density*. As a consequence of the definition of copulas, in the case of absolutely continuous copulas we have the following:

$$\int_0^1 c(u, v)dv = 1, \quad \text{for all } u \text{ in } [0, 1]; \tag{13}$$

$$\int_0^1 c(u, v)du = 1, \quad \text{for all } v \text{ in } [0, 1]; \tag{14}$$

$$\int_{v_1}^{v_2} \int_{u_1}^{u_2} c(u, v)du\,dv \geq 0 \quad \text{for every } u_1 \leq u_2 \text{ and } v_1 \leq v_2 \text{ in } [0, 1], \tag{15}$$

and the diagonal section may be written in terms of the copula density as

$$\delta_C(u) = \int_0^u \int_0^u c(s, t)ds\,dt. \tag{16}$$

In the following section we will present a methodology to build families of absolutely continuous copulas with a given diagonal, with no considerations about being neither Archimedean nor symmetric. Before we proceed, it is important to mention an interesting interpretation of diagonal sections given by Sungur and Yang (1996): a diagonal section δ_C is the distribution function of $\max\{U, V\}$ where U and V are continuous uniform variables on $(0, 1)$ with joint distribution function C. The results in the following section allow to build an absolutely continuous joint distribution function for $\{U, V\}$ different from C but with the same given distribution of $\max\{U, V\}$.

2. Construction of a New Family of Copulas

In this section we will construct a broad family of copulas using a fixed absolutely continuous copula $D(u, v)$. The main idea is to construct families of copulas with given restrictions, such as values of the copula on diagonal sections, or horizontal and vertical sections, even including restrictions on closed subsets, in a few words, to construct copulas with a given *agreement region* with $D(u, v)$. In the particular case $D(u, v)$ happens to be of the Archimedean class, we know from Sungur and Yang (1996) that all the information of the copula is contained in the distribution of $\max\{U, V\}$, and so the following methodology allows to build a new joint distribution for $\{U, V\}$ with the given diagonal section $D(u, u)$ as the agreement region.

Let us consider the function

$$f(x, y) = \sin(x)\sin(y) \quad \text{for } x, y \in [0, 2\pi].$$

Then it is clear that if $0 \leq a < b \leq 2\pi$, we have that

$$\int_a^b \int_0^{2\pi} f(x, y)dx\,dy = 0 \quad \text{and} \quad \int_0^{2\pi} \int_a^b f(x, y)dx\,dy = 0,$$

that is, integrals of f along vertical or horizontal segments are always zero. We also observe that $f(x, y) = 0$ on the border of $[0, 2\pi]^2$. We will use appropriate rescalings of the function $f(x, y)$ in order to construct families of absolutely continuous copulas with given diagonals. In fact we will prove the following:

Theorem 2.1. *Let $D(u, v)$ be an absolutely continuous copula with density $\frac{\partial^2}{\partial u \partial v} D(u, v) = d(u, v)$ which will be assumed to be continuous and positive on $[0, 1]^2$. Let $\delta(u) = D(u, u)$ be the diagonal section of $D(u, v)$. Then there exists a family of absolutely continuous copulas $\mathscr{C}$, not necessarily symmetric even when D is symmetric, such that for every $C \in \mathscr{C}$ if $\delta_C(u) = C(u, u)$, then $\delta_C(u) = \delta(u)$, and for almost every $(u, v) \in (0, 1)^2$ $[\lambda]$, where λ is the Lebesgue measure on $[0, 1]^2$, $C(u, v) \neq D(u, v)$.*

Proof. Let $D(u, v)$ be an absolutely continuous copula with density $d(u, v) = \frac{\partial^2}{\partial u \partial v} D(u, v)$ which is continuous and positive on $[0, 1]^2$. Since $d(u, v)$ is continuous and positive on $[0, 1]^2$, which is compact, then there exists $M > 0$, such that $d(u, v) \geq M$ for every $(u, v) \in [0, 1]^2$.

Let $\Delta_u := \{(u, v) \in [0, 1]^2 \mid u \leq v\}$ and $\Delta_l := \{(u, v) \in [0, 1]^2 \mid u \geq v\}$, be the upper and lower triangles above and below the diagonal of $[0, 1]^2$. Let $0 \leq u_1 < u_2 \leq v_1 < v_2 \leq 1$, then the rectangle $[u_1, u_2] \times [v_1, v_2] \subset \Delta_u$. Similarly, if $0 \leq v_1 < v_2 \leq u_1 < u_2 \leq 1$, then the rectangle $[u_1, u_2] \times [v_1, v_2] \subset \Delta_l$.

Now we rescale the function f given just before this theorem to the rectangle $J = [u_1, u_2] \times [v_1, v_2]$. That is, we consider

$$f_J(u, v) = \sin\left(\frac{2\pi(u - u_1)}{u_2 - u_1}\right) \sin\left(\frac{2\pi(v - v_1)}{v_2 - v_1}\right) 1_J(u, v), \tag{17}$$

where 1_A denotes the indicator function of the set A. If $u_1 \leq u \leq u_2$ and $v_1 \leq v \leq v_2$, then

$$\begin{aligned}
&\int_{v_1}^{v} \int_{u_1}^{u} f_J(s, t) ds\, dt \\
&= \int_{v_1}^{v} \int_{u_1}^{u} \sin\left(\frac{2\pi(s - u_1)}{u_2 - u_1}\right) \sin\left(\frac{2\pi(t - v_1)}{v_2 - v_1}\right) ds\, dt \\
&= \frac{(u_2 - u_1)(v_2 - v_1)}{4\pi^2} \int_0^{\frac{2\pi(u-u_1)}{u_2-u_1}} \int_0^{\frac{2\pi(v-v_1)}{v_2-v_1}} \sin(w) \sin(z) dw\, dz \\
&= \frac{(u_2 - u_1)(v_2 - v_1)}{4\pi^2} \left(1 - \cos\left(\frac{2\pi(u - u_1)}{u_2 - u_1}\right)\right) \left(1 - \cos\left(\frac{2\pi(v - v_1)}{v_2 - v_1}\right)\right).
\end{aligned}$$

Of course this integral is always non negative, and if $u = u_2$ or $v = v_2$, then $\int_{v_1}^{v} \int_{u_1}^{u} f_J(s, t) ds\, dt = 0$. On the other hand, as can be easily verified, if $u = (u_2 + u_1)/2$ and $v = (v_2 + v_1)/2$, then $\int_{v_1}^{v} \int_{u_1}^{u} f_J(s, t) ds\, dt = (u_2 - u_1)(v_2 - v_1)/\pi^2$, which is the maximum value for this integral.

Let $I_{1,1} = [0, 1/2] \times [1/2, 1]$ and $I_{1,2} = [1/2, 1] \times [0, 1/2]$, then $I_{1,1} \subset \Delta_u$ and $I_{1,2} \subset \Delta_l$. For $k \geq 2$ and $j = 1, 2, \ldots, 2^k$, define

$$I_{k,j} := \begin{cases} \left[\frac{j-1}{2^k}, \frac{j}{2^k}\right] \times \left[\frac{j}{2^k}, \frac{j+1}{2^k}\right] & \text{if } j = 1, 3, \ldots, 2^k - 1 \\ \left[\frac{j-1}{2^k}, \frac{j}{2^k}\right] \times \left[\frac{j-2}{2^k}, \frac{j-1}{2^k}\right] & \text{if } j = 2, 4, \ldots, 2^k. \end{cases}$$

Then for every $k \geq 2$ and $j = 1, 2, \ldots, 2^k$, $I_{k,j} \subset \Delta_u$ if j is odd and $I_{k,j} \subset \Delta_l$ if j is even, see Fig. 1, for the cases $k = 1$ and $k = 2$. If we denote the interior of a set A by $\text{int}(A)$, then it is also clear that for any k and every $j_1, j_2 \in \{1, 2, \ldots, 2^k\}$,

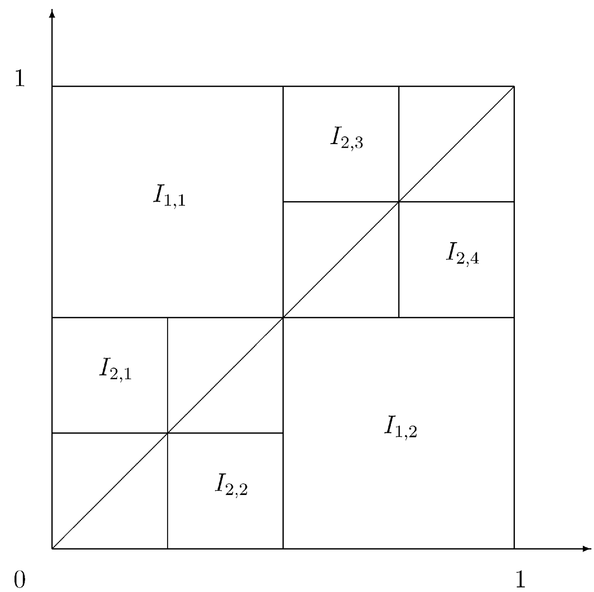

Figure 1. Graph of the $I'_{k,j}s$ for $k = 1, 2$.

with $j_1 \neq j_2$, we have that $\text{int}(I_{k,j_1}) \cap \text{int}(I_{k,j_2}) = \emptyset$. We also have that for any $k < l$ and any $j_1 \in \{1, 2, \ldots, 2^k\}$ and $j_2 \in \{1, 2, \ldots, 2^l\}$, $\text{int}(I_{k,j_1}) \cap \text{int}(I_{l,j_2}) = \emptyset$. We finally observe that

$$\Delta_u = \bigcup_{k=1}^{\infty} \bigcup_{j=1,3,\ldots,2^k-1} I_{k,j} \quad \text{and} \quad \Delta_l = \bigcup_{k=1}^{\infty} \bigcup_{j=2,4,\ldots,2^k} I_{k,j}.$$

Now, for every $k \geq 1$ and every $(u, v) \in [0, 1]^2$ we define using (17)

$$f_{(k;\alpha_{k,1},\ldots,\alpha_{k,2^k})}(u, v) = \sum_{j=1}^{2^k} \alpha_{k,j} f_{I_{k,j}}(u, v) \quad \text{where } |\alpha_{k,j}| \leq M \text{ for every } j = 1, 2, \ldots, 2^k.$$

It is important to observe that for any $0 \leq a < b \leq 1$ and any selection of $\alpha_{k,1}, \ldots, \alpha_{k,2^k}$,

$$\int_0^1 \int_a^b f_{(k;\alpha_{k,1},\ldots,\alpha_{k,2^k})}(u, v)\, du\, dv = \int_a^b \int_0^1 f_{(k;\alpha_{k,1},\ldots,\alpha_{k,2^k})}(u, v)\, du\, dv = 0. \tag{18}$$

This property follows from the definition of $f_{(k,\alpha_{k,1},\ldots,\alpha_{k,2^k})}$ and the definition of the subsets $I_{k,j}$, see Fig. 1. Now for $n \geq 1$ define

$$f_n(u, v) = d(u, v) + \sum_{k=1}^{n} f_{(k;\alpha_{k,1},\ldots,\alpha_{k,2^k})}(u, v),$$

then f_n depends on $\alpha_{1,1}, \alpha_{1,2}, \alpha_{2,1}, \ldots, \alpha_{2,4}, \ldots, \alpha_{n,1}, \ldots, \alpha_{n,2^n}$, that is $2 + 2^2 + \cdots + 2^n = 2^{n+1} - 2$ parameters. From the selection of the $\alpha_{k,j}$, for $k = 1, \ldots, n$ and $j = 1, \ldots, 2^k$, we have that every $f_n(u, v)$ is a non negative continuous function, since $d(u, v)$ is continuous as well as every $f_{(k,\alpha_{k,1},\dots,\alpha_{k,2^k})}$. From Eq. (18) we also obtain that

$$\int_0^1 \int_0^1 f_n(u, v) du\, dv = \int_0^1 \int_0^1 d(u, v) = 1.$$

Hence, for every $n \geq 1$ and selection of parameters $|\alpha_{k,j}| \leq M$, for $k = 1, \dots, n$ and $j = 1, \dots, 2^k$, $f_n(u, v)$ is a continuous density. In fact, it is the density of an absolutely continuous copula whose formula is given by

$$C_n(u, v) = D(u, v) + \sum_{k=1}^{n} \sum_{j=1}^{2^k} \frac{\alpha_{k,j}}{2^{2k+2}\pi^2} \left\{1 - \cos\left(2^{k+1}\pi(u - ((j-1)/2^k))\right)\right\} \times \left\{1 - \cos\left(2^{k+1}\pi(u - (j^*/2^k))\right)\right\} 1_{I_{k,j}}(u, v),$$

where $j^* = j$ if j is odd and $j^* = j - 2$ if j is even. Here we observe that $C_n(u, v)$ is close to $D(u, v)$ except for perturbations on every $I_{k,j}$. In fact, it is easy to see that

$$\sup_{(u,v)\in[0,1]^2} |C_n(u, v) - D(u, v)| \leq \frac{M}{4\pi^2},$$

for every selection of the parameters $\alpha_{k,j}$. Of course, if every $\alpha_{k,j} \neq 0$, then $C_n(u, v) \neq D(u, v)$ for almost every $(u, v) \in I_{k,j}$, except only if $u = (2j - 1)/2^{k+1}$, or if $v = (2j + 1)/2^{k+1}$ with j odd, or $v = (2j - 3)/2^{k+1}$ with j even.

If we define $C(u, v) = \lim_{n\to\infty} C_n(u, v)$, we obtain that $C(u, v)$ is a copula depending on an infinite number of parameters, if all of them are non zero, then $C(u, v) \neq D(u, v)$ almost surely for the Lebesgue measure on $[0, 1]^2$. We finally observe that if $\alpha_{k,j} \neq \alpha_{k,j+1}$ for any $k \geq 1$ and any $j = 1, 3, \dots, 2^k - 1$, then $C_n(u, v)$ is a non symmetric copula for any $n \geq k$. In fact, we can construct, using the methodology above, an almost everywhere asymmetric copula with respect to Lebesgue measure. □

Example 2.1. Let $D(u, v) = uv$ for $(u, v) \in [0, 1]^2$. Then D is an absolutely continuous copula with density $d(u, v) = 1$ for every $(u, v) \in [0, 1]^2$. Let $J = [0, 1/2] \times [1/2, 1]$, if we define

$$C(u, v) = \begin{cases} uv + \dfrac{1}{16\pi^2}\{\cos(4\pi u) - 1\}\{\cos(4\pi(v - 1/2)) - 1\} & \text{if } (u, v) \in J \\ uv & \text{if } (u, v) \in [0, 1]^2 \backslash J. \end{cases}$$

Then $C(u, v)$ is an asymmetric copula which coincides with $D(u, v)$ on the diagonal.

We do have easy extensions of the Theorem 2.1, such as the following.

Corollary 2.1. *Let $D(u, v)$ be an absolutely continuous copula with density $\frac{\partial^2}{\partial u \partial v} D(u, v) = d(u, v)$ which we will be assumed to be continuous and positive on $[0, 1]^2$. Let $n, m \geq 1$ and let $0 = u_0 < u_1 < u_2 < \cdots < u_{n-1} < u_n = 1$ and $0 = v_0 < v_1 < v_2 < \cdots < v_{m-1} < v_m = 1$ any points. Consider the vertical sections $V_k = \{(u, v) \in [0, 1]^2 \mid u = u_k\}$, for $k = 0, \dots, n$ and the horizontal sections $H_j = \{(u, v) \in [0, 1]^2 \mid v = v_j\}$ for $j = 0, \dots, m$. Then there exists an infinite family of copulas $\mathscr{C}$, such that for* *every $C \in \mathscr{C}$, C is absolutely continuous and $C(u, v) = D(u, v)$ for every $(u, v) \in V_k$, $k = 0, 1, \dots, n$ and every $(u, v) \in H_j$, $j = 0, 1, \dots, m$.*

The proof of this corollary follows the same steps as Theorem 2.1, by defining the density of C on every $I_{k,j} = [u_k, u_{k+1}] \times [v_j, v_{j+1}]$, for every $k = 0, 1, \dots, n$ and $j = 0, 1, \dots, m$.

The function $f(u, v) = \sin(u)\sin(v)$ on $[0, 2\pi]^2$ can be substituted by any function of the form $f(u, v) = g(u)g(v)$ on $[a, b]^2$, as long as $a < b$, g is continuous with $g(a) = g(b) = 0$, and $\int_a^b g(x)dx = 0$.

In the above corollary the term "infinite family of copulas" is due to the different ways of defining the density of C on every $I_{k,j} = [u_k, u_{k+1}] \times [v_j, v_{j+1}]$ and the use of all the functions of the form $f(u, v) = g(u)g(v)$ mentioned above.

Another way of proving Theorem 2.1, is to rescale the function

$$f(u, v) = \sin(2\pi v)\sin(2\pi u/v)1_{\{(u,v)\in[0,1]^2 \,|\, u\leq v\}}(u, v),$$

and its symmetric version.

We can also find families of copulas that agree on closed sets with a given absolutely continuous copula. For example, copulas that agree with the absolute continuous copula $D(u, v)$ on $[1/4, 3/4] \times [1/4, 3/4]$, or even on circles such as $\{(u, v) \in [0, 1]^2 \,|\, (u - 1/2)^2 + (v - 1/2)^2 \leq 1/4\}$, simply by noticing that the usual topology on $[0, 1]^2$ with the usual metric $d((x, y), (u, v)) := ((x - u)^2 + (y - v)^2)^{1/2}$ is the same as the topology metrized by $\rho((x, y), (u, v)) := \max\{|x - u|, |y - v|\}$ where the open balls are (open) rectangles.

In fact, from the remark above we have the following.

Corollary 2.2. *Let $D(u, v)$ be an absolutely continuous copula with density $\frac{\partial^2}{\partial u \partial v}D(u, v) = d(u, v)$ which we will be assumed to be continuous and positive on $[0, 1]^2$. Let $C(u, v)$ another copula such that $C(u, v) = D(u, v)$ on a largest closed subset $A \subset [0, 1]^2$. Then $C(u, v) = D(u, v)$ for every $(u, v) \in [0, 1]^2$ if and only if $A = [0, 1]^2$.*

Proof. Assume that $C(u, v) = D(u, v)$ on a closed subset $A \subset [0, 1]^2$, but $A \neq [0, 1]^2$. Then A^c, the complement of A is an open non empty set. Hence, we can find $0 < u_1 < u_2 < 1$ and $0 < v_1 < v_2 < 1$, such that $J = [u_1, u_2] \times [v_1, v_2] \subset A^c$. By defining f on this rectangle as in Theorem 2.1, we obtain a copula which coincides with $D(u, v)$ on A but it is different from D on J. □

From Corollary 2.2, the only way to determine uniquely an absolutely continuous copula is by giving its values on a dense subset of $[0, 1]^2$.

The hypothesis of a copula having a positive density on $[0, 1]^2$, can be also weakened, obtaining similar results. For example, the density can be zero on the border of $[0, 1]^2$, or even the density can be zero on a closed region, and still the construction will work outside this region.

It is important to notice that in all previous results, we can construct asymmetric copulas that follow the given restrictions.

3. Final Remarks

By the results obtained by Sungur and Yang (1996) we know that in the case of Archimedean copulas the values of the diagonal $\delta(u) = C(u, u)$ determine uniquely the behavior of the copula on $[0, 1]^2$.

Outside the Archimedean family of copulas, Fredricks and Nelsen (1997a,b, 2002) provide examples of copulas with certain restrictions, such as the values of the copula on the diagonal. These examples provide singular and symmetric copulas.

In this article, we provide a broad family of absolutely continuous copulas with a fixed diagonal, which can differ from another absolutely continuous copula almost everywhere with respect to Lebesgue Measure. Recalling form Sungur and Yang (1996) that a diagonal section δ_C is the distribution function of $\max\{U, V\}$ where U and V are continuous uniform variables on $(0, 1)$ with joint distribution function C, the obtained results allow to build an absolutely continuous joint distribution function for $\{U, V\}$ different from C but with the same given distribution of $\max\{U, V\}$.

In fact, we can extend these results to a finite number of horizontal and vertical regions, as well as extending them to closed sets, which are not dense on $[0, 1]^2$.

We notice that the asymmetry in the methodology proposed here, is not an issue, since the construction itself allows to find asymmetric copulas.

In the present work we focused on bivariate copulas but it is possible to extend our results for dimensions higher than two without dealing with the *compatibility problem* because the starting point for building, say, an n-dimensional asymmetric and absolutely continuous copula $C(u_1, \ldots, u_n)$ would be a given n-dimensional absolutely continuous copula $D(u_1, \ldots, u_n)$ (see Theorem 2.1) and so the proposed methodology does not deal with the compatibility with $(n-m)$-dimensional marginal copulas, where $2 \leq m < n$. For example, with the analogous ideas used in Example 2.1, define

$$C(u, v, w) = \begin{cases} uvw + \dfrac{1}{64\pi^3}[\cos(4\pi u) - 1]\left[\cos\left(4\pi\left(v - \dfrac{1}{2}\right)\right) - 1\right] \\ \quad [\cos(4\pi w) - 1], & \text{if } (u, v, w) \in J, \\ uvw & \text{if } (u, v, w) \in [0, 1]^3 \backslash J, \end{cases}$$

where $J = \left[0, \frac{1}{2}\right] \times \left[\frac{1}{2}, 1\right] \times \left[0, \frac{1}{2}\right]$. Then $C(u, v, w)$ is an asymmetric copula which coincides with the independence copula $\Pi(u, v, w) = uvw$ on the diagonal.

We can also find families of n-dimensional copulas that agree on closed sets with a given n-dimensional absolutely continuous copula, even on n-dimensional spheres simply by noticing that the usual topology on $[0, 1]^n$ with the usual metric $d(\mathbf{x}, \mathbf{u}) := ((x_1 - u_1)^2 + \cdots + (x_n - u_n)^2)^{1/2}$ is the same as the topology metrized by $\rho(\mathbf{x}, \mathbf{u}) := \max\{|x_1 - u_1|, \ldots, |x_n - u_n|\}$ where the open balls are (open) n-cubes.

In the literature, many authors have worked on fitting Archimedean copulas to real data. However, even the assumption of symmetry alone is too strong, as can be seen from the constructions of copulas with restrictions in this work. We could define some ways to measure asymmetry, for example, given a bivariate copula C:

$$\xi_C := \sup_{(u,v)\in[0,1]^2} |C(u, v) - C(v, u)|,$$

and use its empirical version to try to build a non parametric test of symmetry, before thinking about fitting an Archimedean copula, since symmetry is a necessary (but not sufficient) condition for such copulas. In the case that the test rejects the null hypothesis $H_0 : C(u, v) = C(v, u)$ for all $u, v \in [0, 1]$, this would suggest that an asymmetric copula should be considered for fitting, and there is no such a big list of known asymmetric copulas to choose from, unlike the case of symmetric copulas. The present work is a contribution to the list of families of asymmetric copulas.

As stated by Sungur and Yang (1996), since the diagonal section uniquely determines the copula for the Archimedean class, it simplifies the model building, fitting and checking process. They also notice that, within the class of Archimedean copulas, a goodness of fit test can be carried out in terms of the diagonal section. They mention as an example a test of independence using the fact that the independence bivariate copula $\Pi(u, v) = uv$ is Archimedean: let (X, Y) be a random vector with Archimedean copula C, then the following hypothesis are equivalent:

$$H_0 : X \text{ and } Y \text{ are independent} \Leftrightarrow H_0 : C = \Pi \Leftrightarrow H_0 : \delta_C(u) = u^2.$$

For this purpose, a suitable test statistic is required, and we propose

$$D_n := \max_{j=1,\ldots,n-1} \left| \Delta\left(\frac{j}{n}\right) - \left(\frac{j}{n}\right)^2 \right|,$$

where

$$\Delta\left(\frac{j}{n}\right) := \frac{1}{n} \sum_{k=1}^{j} 1_{\{Y_k \leq Y_{(j)}\}}, \quad j = 1, \ldots, n-1.$$

We have obtained the exact distribution under H_0, (Erdely and González-Barrios, 2005). The assumption of an underlying copula of the Archimedean class is too strong, so one may ask about the power of such test outside the Archimedean class, which in principle leads to the question if there exists an absolutely continuous copula C different from the independence copula Π such that $\delta_C(u) = u^2$. This question motivated the present work and the answer is positive as we have already shown. So one should be careful in using an independence test based on the diagonal outside the Archimedean class.

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