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Frank's condition for multivariate Archimedean copulas

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Abstract

In this paper we study possible multivariate generalizations of construction of n-copulas with a given diagonal section. In the case of Archimedean n-copulas we prove that Frank's condition, which establishes when an Archimedean 2-copula is characterized by its diagonal section, can be extended to larger dimensions.

Keywords: Multivariate copulas; Diagonal sections

1. Introduction

In this paper we give some possible multivariate generalizations of known results for 2-copulas.

Recall that an n-copula is a joint distribution function on $\mathbb{R}^n$ for some $n \geqslant 2$, restricted to $[0,1]^n$ with uniform $(0,1)$ margins.

If we define

$$W^n(u_1,\dots,u_n)=\max\{u_1+\cdots u_n-n+1,0\} \quad \text{and} \quad M^n(u_1,\dots,u_n)=\min\{u_1,\dots,u_n\}$$

for every $\langle u_1,\dots,u_n\rangle \in [0,1]^n$. Then W^n is a copula if and only if $n=2$, and M^n is always an n-copula. However any n-copula C satisfies

$$W^n(u_1,\dots,u_n)\leqslant C(u_1,\dots,u_n)\leqslant M^n(u_1,\dots,u_n) \quad \text{for every } \langle u_1,\dots,u_n\rangle \in [0,1]^n.$$

Let C be an n-copula, then the diagonal section of C, denoted by $\delta_C(t)$, is defined by $\delta_C(t)=C(t,\dots,t)$ for every $t\in[0,1]$. Therefore, if C is an n-copula with *diagonal section* δ_C, then δ_C satisfies the following conditions:

i) $\delta_C:[0,1]\to[0,1]$ is an increasing function, that is, $\delta_C(s)\leqslant\delta_C(t)$ if $0\leqslant s\leqslant t\leqslant 1$, with $\delta_C(0)=0$ and $\delta_C(1)=1$.
ii) $0\leqslant\delta_C(t)-\delta_C(s)\leqslant n(t-s)$ for every $0\leqslant s\leqslant t\leqslant 1$.
iii) $\max\{nt-n+1,0\}\leqslant\delta_C(t)\leqslant t$ for every $t\in[0,1]$.

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If $\delta : [0, 1] \to [0, 1]$ is a function, then δ is called a *diagonal* if and only if δ satisfies conditions i), ii) and iii). A very important result that characterizes diagonal sections of 2-copulas is the following result of Fredricks and Nelsen [9] or Fredricks and Nelsen [10]:

Theorem 1.1. *Let $\delta : [0, 1] \to [0, 1]$ be a diagonal. Then there exists a copula C such that $\delta_C(t) = \delta(t)$ for every $t \in [0, 1]$.*

In particular, in [9] the authors proved that if δ is a diagonal then

$$C(u, v) = \min\left\{u, v, \frac{\delta(u) + \delta(v)}{2}\right\} \quad \text{for every } \langle u, v\rangle \in [0, 1]^2 \tag{1}$$

is a singular copula with diagonal section δ, in fact, C is *symmetric*, that is, $C(u, v) = C(v, u)$ for every $\langle u, v\rangle \in [0, 1]^2$. Another construction of copulas with given diagonal δ can be found in [2], and it is given by

$$B(u, v) = \min\{u, v\} - \inf_{\min\{u,v\} \leqslant t \leqslant \max\{u,v\}} \big[t - \delta(t)\big] \quad \text{for every } \langle u, v\rangle \in [0, 1]^2. \tag{2}$$

Then B is a singular copula called *Bertino* copula with diagonal δ which is also symmetric.

Recall also, that if $\varphi : [0, 1] \to [0, \infty]$ is a strictly decreasing convex function with $\varphi(1) = 0$ and pseudo-inverse $\varphi^{[-1]}$, and we define

$$C(u, v) = \varphi^{[-1]}\big(\varphi(u) + \varphi(v)\big) \quad \text{for every } \langle u, v\rangle \in [0, 1]^2. \tag{3}$$

Then C in (3) is a copula, called *Archimedean copula* with generator φ, see for example [19].

The function φ in Eq. (3) is called an *Archimedean strict generator* if $\varphi(0) = \infty$ and an *Archimedean non-strict generator* if $0 < \varphi(0) < \infty$. In fact, it is easy to see that the pseudo-inverse of φ coincides with the usual inverse if φ is a strict generator. We also know that if φ is a generator of an Archimedean copula C, and we define $\psi = c \cdot \varphi$, where $c > 0$ is a positive constant, then ψ is also a generator of C. It is also clear from Eq. (3), that if C is an Archimedean copula with generator φ and δ_C is its diagonal section then

$$\delta_C(t) = \varphi^{[-1]}\big(2\varphi(t)\big) \quad \text{for every } t \in [0, 1]. \tag{4}$$

In the case that φ is a strict generator we can rewrite Eq. (4) as

$$\varphi\big(\delta_C(t)\big) = K\varphi(t) \quad \text{where } K = 2. \tag{5}$$

Eq. (5) for $K \neq 0, 1$ is known in the literature as the Schröder functional equation, and it was studied first by Schröder [21] to solve iterative functional equations. The problem that was proposed by Darsow and Frank [4] is the following: If we know the diagonal section of an Archimedean copula C what can be said about its generator φ?

Sungur and Yang [22] state that for Archimedean copulas its diagonal section determines uniquely the corresponding copula. However, this result does not hold. In fact, in the same year Frank [8] announced in a report of a Symposium on functional equations the following result:

Theorem 1.2 *(Frank's condition). If C is an Archimedean copula with diagonal section δ_C, and $\delta'_C(1-) = 2$ then C is uniquely determined by its diagonal section.*

In the last theorem $\delta'_C(1-)$ denotes the left derivative of δ_C at $x = 1$. The proof of this result was first included in Alsina et al. [1], and it includes an example to see that if Frank's condition is not satisfied then we can construct different Archimedean copulas with the same diagonal section.

In the second section we analyze if it is possible to generalize some of these results to larger dimensions.

In the last section we prove that Frank's condition can be generalized to larger dimension with appropriate changes.

2. Possible extensions of *n*-copulas with a given diagonal section

As we observed in Section 1, W^n is an n-copula if and only if $n = 2$. However, by Theorem 2.10.13 in Nelsen [19], we know that for every $n \geqslant 3$ and for every $0 \leqslant t \leqslant 1$ and $\underline{\mathbf{t}} = \langle t, \dots, t\rangle$, there exists an n-copula C^t such that

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$$\delta_{C^t}(t) = C^t(\underline{\mathbf{t}}) = W^n(\underline{\mathbf{t}}) = \max\{nt - n + 1, 0\}.$$

Therefore, the inequality iii) is sharp.

The following result is well known.

Theorem A. *Let* $\delta : [0, 1]^n \to [0, 1]$ *be a* diagonal *for some* $n \geqslant 2$*, that is,* δ *satisfies conditions* i), ii) *and* iii)*. Then there exists* $C : [0, 1]^n \to [0, 1]$ *an* n*-copula, such that the diagonal section of* C*,* $\delta_C = \delta$*.*

A sketch of the proof of this result is given in Cuculescu and Theoderescu [3], see Rychlik [20] and Jaworski [11]. See also Jaworski and Rychlik [12] and Mesiar and Navara [17].

Remark 2.1. Eqs. (1) and (2) in the introduction cannot be generalized to higher dimensions. We start with Eq. (1) of Fredricks and Nelsen [9] and $n = 3$. Let $\delta : [0, 1] \to [0, 1]$ satisfies equations i), ii) and iii), a possible generalization for this case could be:

$$C(u, v, w) = \min\left\{u, v, w, \frac{\delta(u) + \delta(v) + \delta(w)}{3}\right\} \quad \text{for every } \langle u, v, w\rangle \in [0, 1]^3. \tag{6}$$

$$B(u, v, w) = \min\{u, v, w\} - \inf_{s \in [\min\{u,v,w\}, \max\{u,v,w\}]} \big[s - \delta(s)\big], \tag{7}$$

for Eq. (2). In a more recent paper Durante et al. [6] proposed another copula with a given diagonal. If we try to generalize this to dimension 3, we would have

$$D(x, y, z) = \min\{x, y, z\} - \min\big\{\delta(x) - x, \delta(y) - y, \delta(z) - z\big\}. \tag{8}$$

In the three cases (6), (7) and (8) if we let $R = [0, t] \times [t, 1] \times [t, 1]$ we may obtain negative volumes for some $t \in [0, 1]$ if $\delta \neq \delta_{M^3}$.

3. Frank's condition for Archimedean n-copulas

We start this section by recalling some results about multivariate Archimedean copulas, we start with a basic definition.

Definition 3.1. Let $g : J \to \mathbb{R}$ be an infinitely differentiable function where J is an interval in $\mathbb{R}$. We will say that g is *completely monotonic* if its derivatives alternate in signs, that is, g satisfies

$$(-1)^k \frac{d^k}{dt^k} g(t) \geqslant 0 \tag{9}$$

for every t in the interior of J and for every $k \in \{0, 1, 2, \dots\}$.

Observe that from Eq. (9), a completely monotonic function satisfies that $\frac{d^2}{dt^2} g(t) \geqslant 0$, then g is a convex function, see for example Dudley [5]. Using a result in Widder [23], if g is completely monotonic on $J = [0, \infty)$ and $g(c) = 0$ for some $c \in J$, then g is identically zero on J. Therefore, if the pseudo-inverse $\varphi^{[-1]}$ of an Archimedean generator φ is completely monotonic, it has to be positive on $[0, \infty)$, and so, φ is a strict generator, that is, $\varphi^{[-1]} = \varphi^{-1}$ the usual inverse.

Example 3.2. Let $f(x) = -\ln(x)$ for $0 < x \leqslant 1$, since f is a nonnegative function such that, $f^{(k)}(x) = (-1)^k / x^k$ for every $k \geqslant 1$, then f is completely monotonic according to Definition 3.1. Using that the product of completely monotonic functions is also completely monotonic, see for example Miller and Samko [18], we have that $g(x) = (-\ln(x))^n$ for $0 < x \leqslant 1$ is completely monotonic for every $n \geqslant 2$.

The following theorem was first proved in Kimberling [13], and can be found also in Nelsen [19] and Alsina et al. [1].

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Theorem 3.3. *Let φ be a strict Archimedean generator and let any $n \geqslant 2$. The function $C:[0,1]^n \to [0,1]$ defined by*

$$C(u_1,\dots,u_n) = \varphi^{[-1]}\big(\varphi(u_1) + \cdots + \varphi(u_n)\big) \tag{10}$$

is an n-copula if and only if φ^{-1} is completely monotonic.

It is clear that if we define $\varphi(t) = -\ln(t)$ then $\varphi^{-1}(t) = \exp(-t)$ which is clearly completely monotonic, besides we know that φ is the generator of the product copula $\Pi(u_1,\dots,u_n) = u_1 \cdot u_2 \cdot \dots \cdot u_n$, see Nelsen [19].

More recently, in McNeil and Nešlehová [16], they studied necessary and sufficient conditions for a generator to construct d-dimensional copulas, for a fixed $d \geqslant 2$, one of their results states:

Theorem B. *Let φ be a strict Archimedean generator with inverse $\phi = \varphi^{-1}$ which has derivatives up to order d on $(0,1)$. Then φ generates an Archimedean d-copula if and only if $(-1)^k\phi^{(k)}(x) \geqslant 0$ for $k = 0,1,\dots,d$.*

This result is Corollary 2.1 in McNeil and Nešlehová [16].

The following result is a particular case of Theorem 6.6 in Kuczma [14], or Theorem 2.3.12 in Kuczma et al. [15].

Theorem 3.4. *Let $\gamma:[0,1] \to [0,1]$ be a function such that $0 < \gamma(u) < u$ for every $u \in (0,1)$, and assume that $\gamma'(0+) = \frac{1}{n}$ for some $n \geqslant 2$ fixed. If $s(u)$ is a solution of the functional equation*

$$s\big(\gamma(u)\big) = \frac{1}{n}s(u) \tag{11}$$

such that $s(u)/u$ is monotonic in $(0,1)$, then

$$s(u) = k \lim_{m\to\infty} n^m \gamma^m(u) \tag{12}$$

where γ^m is the m-th iteration of γ, that is, the composition of γ with itself m times, and k is any positive constant. Solution (12) is continuous, convex, unique (up to multiplicative constants) and for $k > 0$ strictly monotonic in $[0,1]$.

Now we will see that Frank's condition in Alsina et al. [1] can be extended to n-Archimedean copulas using the last Theorem.

Theorem 3.5. *Let $n \geqslant 3$ and let C be an n-Archimedean copula whose diagonal δ_C satisfies $\delta_C'(1-) = n$. Then C is uniquely determined by its diagonal.*

Proof. Let φ the strict generator of an n-Archimedean copula C for some $n \geqslant 3$, then using Eq. (10), we have that its diagonal satisfies that $\delta(u) = \varphi^{-1}(n\varphi(u))$ for every $u \in [0,1]$, which is a continuous and strictly increasing function. Therefore, its inverse δ^{-1} is well defined. Now, take $\gamma:[0,1] \to \mathbb{R}$ defined by $\gamma(u) = 1 - \delta^{-1}(1-u)$ for every $u \in [0,1]$, then γ is continuous, strictly increasing, with $\gamma(0) = 0$ and $0 < \gamma(u) < u$ for every $u \in (0,1)$. If we define $s(u) = \varphi(1-u)$ and substituting the definition of γ and s in Schröder's functional equation we have that $\varphi(\delta(u)) = n\varphi(u)$, we get that this functional equation is equivalent to Eq. (11). So, by requiring that $s(u)/u$ be monotonic and $\lim_{u\to 0}[\gamma(u)/u] = 1/n$ we can apply Theorem 3.4, since the existence of a solution $s(u)$ in (11) is guaranteed by the existence of $\varphi(u)$ as a consequence of the Archimedeanity hypothesis. The last condition is fulfilled if γ is right differentiable in zero and $\gamma'(0+) = \frac{1}{n}$ which is equivalent to $\delta'(1-) = n$. Besides, since the generator φ is convex and so it is $s(u) = \varphi(1-u)$, then by Proposition 6.3.2 in Dudley [5], we have that $s(u)/u$ is monotonic. Applying Theorem 3.4, we obtain the following formula for φ in terms of the diagonal δ.

$$\varphi(u) = k \lim_{m\to\infty} n^m\big[1 - \delta^{-m}(u)\big], \tag{13}$$

where δ^{-m} is the composition of δ^{-1} with itself m times and k is a positive constant, because we require that $\varphi \geqslant 0$. Hence, from Theorem 3.4 solution (13) is unique (up to multiplicative constants which generate the same copula C). □

In Alsina et al. [1, Section 3.8], a counterexample is given, in order to show that if $n = 2$ and φ is generator for an Archimedean copula C such that $\varphi'(1-) = 0$, or equivalently $\delta_C(1-) < 2$, where δ_C is the diagonal of the copula C, then the diagonal does not characterize uniquely the generator φ. Alsina et al. [1] provide a parametric family of generators $\{\varphi_{\beta,2} \mid 0 \leqslant \beta \leqslant 1/(1+8\pi)\}$ such that the diagonal section $\delta_{\beta_1,2} = \delta_{\beta_2,2} = \delta_C$, but $C_{\beta_1,2} \neq C_{\beta_2,2}$ for $\beta_1 \neq \beta_2$. We will see that their upper bound for the values of β can be improved. They define for $0 \leqslant \beta \leqslant 1$ and for $0 < x < 1$

$$\varphi_{\beta,2}(x) = \big(\ln(x)\big)^2 + 2^n \beta \sin\left(\frac{(\ln(x))^2}{2^n}\right) \quad \text{if} \quad 2^{n+1}\pi \leqslant \big(\ln(x)\big)^2 \leqslant 2^{n+2}\pi, \tag{14}$$

for $n \in \mathbb{Z}$. Observe that the first term in function $\varphi_{\beta,2}$ in Eq. (14) corresponds to the generator of the Gumbel–Hougaard family with $\theta = 2$, see Nelsen [19, Section 4.2]. The second term in Eq. (14) is just a small perturbation of the generator by a periodic function. We also observe that the condition $2^{n+1}\pi \leqslant (\ln(x))^2 \leqslant 2^{n+2}\pi$ is equivalent to

$$\exp\big(-\sqrt{2^{n+2}\pi}\,\big) \leqslant x \leqslant \exp\big(-\sqrt{2^{n+1}\pi}\,\big),$$

then we have that

$$\lim_{n\to\infty} \exp\big(-\sqrt{2^{n+2}\pi}\,\big) = 0 \quad \text{and} \quad \lim_{n\to-\infty} \exp\big(-\sqrt{2^{n+2}\pi}\,\big) = 1.$$

Then $\lim_{x\to 0} \varphi_{\beta,2}(x) = \infty$ and $\lim_{x\to 1} \varphi_{\beta,2}(x) = 0$. It is easy to see that $\varphi_{\beta,2}$ in Eq. (14) is twice differentiable, where

$$\varphi'_{\beta,2}(x) = \frac{2\ln(x)}{x}\left(1 + \beta\cos\left(\frac{(\ln(x))^2}{2^n}\right)\right),$$

and

$$\varphi''_{\beta,2}(x) = \frac{2(1-\ln(x))}{x^2}\left(1 + \beta\cos\left(\frac{(\ln(x))^2}{2^n}\right)\right) - \frac{4(\ln(x))^2}{2^n x^2}\beta\sin\left(\frac{(\ln(x))^2}{2^n}\right),$$

for $\exp(-\sqrt{2^{n+2}\pi}\,) \leqslant x \leqslant \exp(-\sqrt{2^{n+1}\pi}\,)$ and for every $n \in \mathbb{Z}$. Evaluating numerically the first and second derivatives of $\varphi_{\beta,2}$ we observed that $\varphi'_{\beta,2}(x) \leqslant 0$ and $\varphi''_{\beta,2}(x) \geqslant 0$ if and only if $0 < \beta \leqslant 0.062548$, and this value is sharp. However, the upper bound of β in Alsina et al. [1] is $1/(1+8\pi) = 0.038266$ is not sharp, see [7].

Now we give a counterexample, in order to show that if $n = 3$ and φ is generator for an Archimedean 3-copula C such that $\varphi'(1-) = 0$, or equivalently $\delta_C(1-) < 3$, where δ_C is the diagonal of the 3-copula C, then the diagonal does not characterize uniquely the generator φ. We provide a parametric family of generators $\{\varphi_{\beta,3} \mid 0 \leqslant \beta \leqslant K\}$ for some $K > 0$, such that the diagonal section $\delta_{\beta_1,3} = \delta_{\beta-2,3} = \delta_C$, but $C_{\beta_1,3} \neq C_{\beta_2,3}$ for $\beta_1 \neq \beta_2$. We define for $0 \leqslant \beta \leqslant 1$ and for $0 < x < 1$

$$\varphi_{\beta,3}(x) = \big(\ln(x)\big)^4 + 3^m \beta \sin\left(\frac{(\ln(x))^4}{3^m}\right) \quad \text{if } 3^{m+1}\pi \leqslant \big(\ln(x)\big)^4 \leqslant 3^{m+2}\pi, \tag{15}$$

for $m \in \mathbb{Z}$. Again, the first term in function $\varphi_{\beta,3}$ in Eq. (15) corresponds to the generator of the Gumbel–Hougaard family with $\theta = 4$, and the second term in Eq. (15) is just a small perturbation of the generator by a periodic function. In this case the condition $3^{m+1}\pi \leqslant (\ln(x))^4 \leqslant 3^{m+2}\pi$ for $m \in \mathbb{Z}$, is equivalent to

$$\exp\big(-\big(3^{m+2}\pi\big)^{1/4}\big) \leqslant x \leqslant \exp\big(-\big(3^{m+1}\pi\big)^{1/4}\big),$$

then we have that

$$\lim_{m\to\infty} \exp\big(-\big(3^{m+2}\pi\big)^{1/4}\big) = 0 \quad \text{and} \quad \lim_{m\to-\infty} \exp\big(-\big(3^{m+2}\pi\big)^{1/4}\big) = 1.$$

Then $\lim_{x\to 0} \varphi_{\beta,3}(x) = \infty$ and $\lim_{x\to 1} \varphi_{\beta,3}(x) = 0$. It is easy to see that $\varphi_{\beta,3}$ in Eq. (15) is continuous, and in fact, it is three times differentiable, where

$$\varphi'_{\beta,3}(x) = \frac{4(\ln(x))^3}{x}\left(1 + \beta\cos\left(\frac{(\ln(x))^4}{3^m}\right)\right),$$

$$\varphi''_{\beta,3}(x) = \frac{4(\ln(x))^2[3-\ln(x)]}{x^2}\left(1 + \beta\cos\left(\frac{(\ln(x))^4}{3^m}\right)\right) - \frac{16(\ln(x))^6}{3^m x^2}\beta\sin\left(\frac{(\ln(x))^4}{3^m}\right),$$

and

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$$\varphi_{\beta,3}'''(x) = \frac{24\ln(x) - 36(\ln(x))^2 + 8(\ln(x))^3}{x^3}\left(1 + \beta\cos\left(\frac{(\ln(x))^4}{3^m}\right)\right) - \frac{48(\ln(x))^5[3 - \ln(x)]}{3^m x^3}\beta\sin\left(\frac{(\ln(x))^4}{3^m}\right) - \frac{64(\ln(x))^9}{3^{2m}x^3}\beta\cos\left(\frac{(\ln(x))^4}{3^m}\right). \tag{16}$$

Evaluating numerically it is easy to see that $\varphi_{\beta,3}(x) > 0$ and $\varphi_{\beta,3}'(x) < 0$ for every $0 < x < 1$ and for every $0 \leqslant \beta \leqslant 1$. Besides, $\varphi_{\beta,3}''(x) > 0$ for every $0 < x < 1$ if $0 \leqslant \beta \leqslant 0.0115$. Let $\phi_{\beta,3} = \varphi_{\beta,3}^{-1}$, then using the formulas for the differentiation of inverse functions we know that

$$\phi_{\beta,3}'(x) = \frac{1}{\varphi'(\phi(x))} \quad \text{and} \quad \phi_{\beta,3}''(x) = \frac{-\varphi''(\phi(x))}{[\varphi'(\phi(x))]^2}.$$

Therefore, $\phi_{\beta,3}(x) > 0$ and $\phi_{\beta,3}'(x) < 0$ for every $0 < x < \infty$ and for every $0 \leqslant \beta \leqslant 1$. We also have that $\phi_{\beta,3}''(x) > 0$ for every $0 < x < \infty$ if $0 \leqslant \beta \leqslant 0.0115$. Now for the third derivative of $\phi_{\beta,3}$ we have that

$$\phi_{\beta,3}'''(x) = -\frac{\varphi_{\beta,3}'''(\phi_{\beta,3}(x))}{[\varphi_{\beta,3}'(\phi_{\beta,3}(x))]^4} + \frac{3(\varphi_{\beta,3}''(\phi_{\beta,3}(x)))^2}{[\varphi_{\beta,3}'(\phi_{\beta,3}(x))]^5}. \tag{17}$$

Observe that in this case the sign of the third derivative of $\phi_{\beta,3}$ is not necessarily the sign of $\varphi_{\beta,3}'''$. Evaluating numerically $\phi_{\beta,3}'''(x)$ in Eq. (17), we obtain that $\phi_{\beta,3}'''(x) < 0$ for every $0 < x < \infty$ if $0 \leqslant \beta \leqslant 0.0001267$. Therefore, using Theorem B, if we define $K = 0.0001267$ then the family of generators $\{\varphi_{\beta,3} \mid 0 \leqslant \beta \leqslant K\}$ satisfies the desired conditions.

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