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Bernstein copula modeling for 2D discrete fracture network simulations

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ABSTRACT

In many naturally fractured reservoirs, fractures play a crucial role in their flow and transport properties. An approach that has recently gained popularity for modeling fracture systems is the Discrete Fracture Network (DFN) model. This approach consists in applying a stochastic boolean simulation method, also known as object simulation method, where fractures are represented as simplified geometric objects (line segments in 2D and polygons in 3D). One of the shortcomings of this approach is that it usually does not consider the dependence relationships that may exist between geometric properties of fractures (direction, length, aperture), that is, each property is simulated independently.

In this work a method for modeling such dependencies by copula theory is introduced. In particular, a nonparametric model using Bernstein copulas for direction-length fracture dependency in 2D is presented.

The application of this method is illustrated in an example which shows high agreement between data and simulation, both graphically and in its descriptive statistics, both marginally and jointly, and in the DFN as a whole.

1. Introduction

Fracture network modeling is an important step in reservoir characterization because fractures can be a barrier, or they can be connected so that they form a flow path –in which case permeability estimation is one of the goals of accurate fracture modeling.

Thanks to the increasing computational power, the Discrete Fracture Network (DFN) approach is becoming more feasible. This approach consists in applying an object-based stochastic simulation method, also known as boolean simulation method (Stoyan et al., 1987; Cacas et al., 2001; Chilès and Delfiner, 2012), where fractures are represented as simplified geometric objects (line segments in 2D).

The underlying statistical dependence among natural fracture properties is usually nonlinear and complex; therefore, traditional statistical techniques based on assumptions of linearity are too restrictive for modeling these dependence relationships. For example, it has been reported (Balankin et al., 2001; Olson, 2007) that fracture aperture and length follow heavy tail distributions which usually results in highly nonlinear dependencies not only between them but also with other variables, such as fracture direction. In particular, the relation direction-length is not usually considered because of lack of suitable models but it is important within a dynamic context. In some special cases, when the underlying theory assumption holds, transformations can be applied to reduce the existing dependence to an approximately linear one, but always at the costs of a back-transformation bias (Seber and Wild, 2003; Miller, 1984; Box, 1971). Transformations also pose the additional effort of selecting the transformation more suitable for the dataset, and validating that the underlying theory assumptions hold.

It is common practice to simplify the dependence among random variables to a scalar number: a correlation coefficient. The problem with this oversimplification is that two datasets could have the same correlation coefficient value, but their dependence structure could be quite different (Kat, 2003; Embrechts et al., 1999; King, 1986; Chernih et al., 2007).

Copula theory has shown to be a very flexible framework to model general dependencies, and this is the approach we use in this work. One of the advantages of copula approach is that univariate marginal distributions and the dependence relationship can be modeled separately (Sklar, 1959).

A valid copula function is an approximation of the empirical copula by Bernstein polynomials (Deheuvels, 1979; Sancetta and Satchell, 2004). This nonparametric approach is simple to use, is almost

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independent of the type of dependence relation, and has shown to reproduce a broad spectrum of dependencies.

One of the shortcomings of Bernstein copulas is that they are not naturally periodic as fracture directions are, but Carnicero et al. (2013) adapted this approach to take into account directional data.

Although Bernstein copulas have been applied for non-directional dependencies (Hernández-Maldonado et al., 2014, 2012; Erdely and Díaz-Viera, 2009, 2012), there is no previous work on explicitly modeling fracture direction-length dependencies in the petroleum industry.

In this work, we show and extend a methodology to model and simulate fracture direction-length relationships, which in turn allows more realistic discrete fracture models.

The application of this method is illustrated in an example which shows high agreement between data and simulation. The univariate nonparametric approach to model the marginal distribution of direction mimics pretty well the periodic behavior of fracture directions, and even shows the skewed parts in the rose diagram. For lengths, it is also observed a coherent match between data and simulation. The Bernstein copula approach reproduces tightly the dependence structure of direction and length. There is also good correspondence of the DFN and the simulated DFN. On the contrary, results using the standard methodology fails to reproduce the expected bivariate behavior.

2. Fracture networks in a geological and petrophysical modeling framework for reservoir characterization

As a first approach, every reservoir should always be treated as fractured (Nelson, 2001).

2.1. Overview of the general integral methodology

The geological and petrophysical modeling of reservoirs, also known as static reservoir characterization, aims to describe and quantify the geometry of the reservoir depositional environment, its structure and stratigraphic relationships, rock types and their property distributions. A proper geological petrophysical model is essential to subsequent tasks such as fluid flow simulation, reserve estimation, and production optimization. There exists a well-established methodology to develop such models in the published literature. In Cosentino (2001) a general integrated methodology is described, while in Deutsch (2002) and Caers (2005) the application of geostatistical methods for reservoir characterization are presented.

A geological and petrophysical model typically involves two stages. Firstly, it is established a quantitative geological model, and subsequently a petrophysical model is built based on the former. Below, there is a brief description of the most important methodological aspects that are considered in geological and petrophysical modeling (Casar-González et al., 2012; Díaz-Viera et al., 2012).

Geological models take the following stages: the structural, stratigraphic, lithological and (optionally) reservoir heterogeneities. Some authors (Nelson, 2001) suggests that the last stage (reservoir heterogeneities) must be mandatory too.

The **Structural model** is the identification of the basic geometric framework of the hydrocarbon trap; it includes the definition of the boundaries. The **stratigraphic model** is responsible for the definition of the internal structure of the reservoir; it is about the construction of a stratigraphic grid and the definition of the main reservoir flow units. The **lithological model** consists in the classification or grouping of facies types in lithotypes or petrophysical classes.

Reservoir heterogeneities are small scale geological features that are considered to have a significant impact on the fluid flow. Examples of such heterogeneities are vugs and fractures. In particular, in naturally fractured carbonate reservoirs, fractures play a major role since they are assumed to affect the main flow path.

The aim of the petrophysical model basically is to populate each facies type with their corresponding petrophysical properties so as to reproduce their representative statistical characteristics. Of course, reservoir heterogeneities must be considered in this step.

Currently, there are a variety of geostatistical stochastic simulation models for reservoir characterization (Deutsch, 2002). The most commonly used models are of two types:

1. **Cell-based models:** A variable is considered to be simulated as a realization of a continuous random function. Examples of variables modeled as continuous random function include porosity and fracture intensity.
2. **Object-based (or boolean) models:** These models can also be considered as marked point processes, in the sense that they are based on a point process and marks are the objects attached to the points of the process (Stoyan et al., 1987; Chilès and Delfiner, 2012). It is also considered as a boolean random set model which corresponds to the intuitive idea of the union of randomly located objects. Many geological features/structures can be modeled using this approach considering them as simplified geometric objects. For example, channels can be represented by sinusoids, lobes by half-ellipsoids, and fractures by line segments in 2D and polygons in 3D.

2.2. Discrete fracture network modeling for fractured reservoirs

Fractures are very important because they can enhance or block the fluid flow (Nelson, 2001; Casar-González et al., 2014, Appendix A). Depending on the fluid flow directions, if fractures are open and connected, they enhance the fluid flow. Otherwise, if closed, fluid cannot pass through them. Therefore, fractures are always important to be considered because of their influence in the dynamic nature of the reservoir.

Notice that the common approach to reservoir characterization (see section 2.1) is complemented by this section, for instance, fractures can be related to facies or structures. Some facies are more prone to fracking than others.

Although there exists several approaches (mainly continuum and discrete) to model and simulate fracture networks (Dershowitz et al., 2004; Dowd et al., 2007; Bonneau et al., 2013; Chilès, 2005; Jing, 2003), the Discrete fracture network (DFN) approach is the most common approach to represent fracture networks in porous media, and particularly, the most used approach in Naturally Fractured Reservoirs.

Discrete fracture network approach considers fractures as simplified geometric objects of lower dimension, usually line segments in 2D and polygons in 3D. Notice that this approach explicitly represents individual fractures.

The 2D case is important because usually the available information to characterize fracture systems is in this way. For example, outcrops, thin films, satellite image data. In this case, and in this paper, individual fractures are modeled as line segments and therefore fracture networks are simplified as networks of line segments.

DFNs are a particular example of the stochastic boolean model (see section 2.1) which is made up of spatial point locations (Poisson germs) on which simplified geometric objects are implanted (Lantuéjoul, 2002, chap. 13). This model allows to independently model a) the spatial concentration of fractures which is measured by its *intensity* (mean fractures per unit area), and b) the simplified fracture properties (direction, length). Due to the appealing characteristics of the DFN approach, it is used by many scientists and implemented in software such as Fracman (Golder associates), Petrel (Schlumberger), FracaFlow (Beicip Franlab), and dfnWorks (los Alamos National Laboratory).

Notice that we have adopted a probabilistic approach since it is usual that the data available in the reservoir is not exhaustive, i.e. there is no perfect knowledge of the fractures in the study area. Fracture characterization and simulation can then be made through the statistical analysis of the available data or by geological models.

The objects may not be deterministic, so they could also be probabilistically modeled. In the latter case, each fracture is generated stochastically so that their spatial and geometric properties are given by probability distribution functions.

The common approach is that fracture locations are independent of each other, i. e., it is assumed a uniform spatial distribution. Also, it is assumed that fracture properties are independent among them (Elmo et al., 2014; Bourbiaux et al., 2002; Zellou et al., 2003; Gringarten, 1997; Adler and Thovert, 1999), therefore, no joint distribution is modeled.

In summary, a real fracture system of a Naturally Fractured Reservoir is modeled as a network of simplified discrete fractures, each fracture having its own direction, length and position.

3. A methodology for discrete fracture network modeling

In a discrete fracture network model using object-based stochastic simulation approach fractures are usually represented as line segments in 2D and as polygons in 3D. Formally, it is convenient to divide the object-based fracture model in three parts: spatial, directional and linear. The *spatial* part consists in the distribution of points where are fractures localized and it is modeled by a stochastic point process. While, the *directional* part consist in the fracture orientation angles and are modeled as directional random variables. And, the *linear* part are the fracture geometric dimensions, such as, length, width and aperture, which are modeled as usual random variables.

In general, any applied statistical modeling requires an Exploratory Data Analysis (EDA) to obtain the basic statistical properties of data, which is often carried out combining visual and quantitative methods. Any EDA consists mainly of plots and descriptive statistics, for example, histograms, scatterplots, sample mean, skewness and correlation coefficients, to mention a few. This preliminar study is very important since it allows to understand the data behavior and consequently to propose the most appropriate model for them.

Each fracture property differs in nature; hence, each one requires a specific analysis. For example, fractures locations, which give intensity, can be analyzed with the theory of point process and with geostatistical tools; fractures directions are analyzed with directional statistics; fracture length, aperture and other attributes, such as porosity, are studied with usual statistics.

The common approach is to analyze each variable independently: univariate exploratory data analysis. Sometimes joint (multivariate) exploratory data analysis is carried out but usually no multivariate model is established, nor joint simulation is performed. This paper offers a methodology to fill this gap for the non-spatial properties: direction and length.

The workflow for each part of the object-based model is described below.

3.1. Spatial

To analyze the spatial position of fractures, a measure of fracture location and concentration is needed. Although there are several such measures for 2D fractures (Nelson, 2001), and within the boolean framework, the geometrical centers of the line segments are commonly used as their coordinate positions. These midpoints are also adopted to this paper.

The statistical tool to analyze spatial point locations is the theory of Point Process (Lantuéjoul, 2002). Within such theory, the test of Complete Spatial Randomness (CSR) is the most basic one, and it attempts to assess if the points have the same probability to occur in each spatial location.

The theory of point process also allows to test if the fractures centers can be modeled with a uniform (CSR), regular, or cluster point process. If uniform, it is also called homogeneous Poisson process. An algorithm to generate simulations can be consulted in Lantuéjoul (2002).

Although points are discrete entities, their concentration is assumed to be a continuous positive (maybe random) function called *intensity*, which is the mean number of points per unit area.

When the CSR test is rejected, or in some cases when it is not clearly rejected, geostatistical analyses have shown that are still useful tools to understand the behavior of fracture intensity.

To use geostatistics, the idea is to think of the fracture intensity as a realization of a continuous random function (Díaz-Viera et al., 2013). In a grid, an estimate of fracture intensity is made by means of box-counting, i. e. counting the number of fractures per grid cell. As a result, a dataset of number of fractures per unit area is obtained. Once again, an EDA must be carried out over the fracture intensity data in order to check if the grid is adequate.

Since the spatial aspect of the DFN is not the focus of this paper, it will be used a homogeneous Poison process.

3.2. Directional

Direction is another property of fractures. From a probabilistic viewpoint, this kind of data can be analyzed with directional statistics. In the case of single directions Θ, the data is called *circular* when in $[0, 360)$, and axial when in the interval $[0, 180)$ (Mardia and Jupp, 2000; Jammalamadaka and SenGupta, 2001; Fisher, 1995).

> Circular data arise in various ways. The two main ways correspond to the two principal circular measuring instruments, the compass and the clock … Data of a similar type arise as times of year (or times of month) of appropriate events.
>
> Mardia and Jupp (2000).

i.e. data with a periodicity p can be analyzed as circular data by linearly rescaling its range to fit in the interval $[0, 2\pi)$. Some authors (Mardia and Jupp, 2000, sec. 1.1; Fisher, 1995) suggest doubling the angles for 2D axial data, as fracture strike, in order to apply the circular statistics theory for analysis, modeling and simulation. Finally, the generated results can be back-transformed to their original range.

All circular random variables must satisfy the same properties of linear random variables (Casella and Berger, 2002), but besides must satisfy the periodicity constraint on its probability density function f:

$$f(\theta) = f(\theta + 2\pi), \quad \theta \in [0, 2\pi) \tag{1}$$

The natural statistical model for the distribution function of directions is the von Mises distribution which can be thought of as similar to the normal distribution: unimodal, symmetric, with one parameter for concentration/dispersion and one for the mean.

Notice that rescaling the 'directional' variable actual range to $[0, 2\pi)$ is a required artifact because the directional statistics theory was developed for such range. Besides, there is a small number of models for circular distribution functions. From these models, frequently the only model computationally available is unimodal and symmetric (von Mises), which constraint further the scope of its usage.

In this work it is used a univariate nonparametric approach (Muñoz Pérez and Fernández-Palacín, 1987) to model fracture direction that does not require any data rescaling, is data-driven, can model non-symmetric data, is easier to fit and simulate than parametric models, and can mimic a much wider spectrum of dataset behavior than the few directional parametric models.

One limitation of using the approach of Muñoz Pérez and Fernández-Palacín (1987) is that is not naturally periodic. Hence, it can only be satisfactorily used in cases where there exists enough data around the limits of the data range. The usage in this paper is justified by assuming enough direction values greater than, but close 0°; and lower than, but close to 180°. For this approach to completely respect the range of the dataset, it is used the directional version of the empirical distribution function (Mardia and Jupp, 2000, p. 100).

3.3. Linear

Fracture length is a linear random variable. While directional data ranges in $[0, 2\pi)$, linear data is defined in the real line $\mathbb{R}$.

In particular, common distribution functions for fracture length are right skewed such as lognormal, power law, Weibull, exponential, gamma and Pareto distributions (Bonnet et al., 2001; Bour and Davy, 1997; Gudmundsson et al., 2011). Although these distributions are parametric, nonparametric length distributions can be used, too. In section 4.4 it is used the nonparametric Bernstein-Kantorovich quantile function (Muñoz Pérez and Fernández-Palacín, 1987).

Other linear variables, which can be also fracture properties, include aperture, porosity and permeability, and are assigned as an attribute (not graphically) to the fracture.

3.4. Directional-linear dependence modeling

Although several (mostly deterministic) models have been proposed for the relation fracture length-aperture (Bonnet et al., 2001, sec 6.4.2, eq. (25); Klimczak et al., 2010; Olson, 2003) none of them provide a dependence measure. To solve this gap, it could be used a circular-linear measure of dependence between direction and length (Kempter et al., 2012; Johnson and Wehrly, 1978; Mardia and Sutton, 1978; Mardia, 1976).

For circular-linear data there exists linear correlation coefficients (Kempter et al., 2012; Johnson and Wehrly, 1977, 1978) similar to Pearson's r product-moment correlation coefficient, but not all relationships are linear, and this assumption can be very restrictive. On the other hand, analogous to Spearman's rank correlation coefficient ρ, Mardia (1976) suggests a nonparametric (rank) correlation coefficient U_n that

> … it is invariant under homeomorphisms (continuous transformations with continuous inverses) of the line and of the circle. An important consequence is that, under independence of X and Θ, the distribution of U_n does not depend on the marginal distributions of X and Θ. The test of independence based on U_n rejects independence for large values of U_n.
>
> Mardia and Jupp (2000, chap. 5).

To evaluate and compare quantitatively some results, a scaled version of U_n that lies in $[0, 1]$, ρ_M (Mardia and Jupp, 2000, p. 247) will be used.

Although correlation coefficients are sometimes useful, it is also well documented the restrictive nature of correlation coefficients to capture the whole dependence structure between random variables.

In this context, the classical option for circular-linear dependence modeling would be an estimation approach using usual regression; but it is well known that regression underestimates variance and extreme values of data. Linear regression for linear-linear data is the most widely known regression technique but usually requires data to be transformed, which introduces bias to the results when back-transformed (Seber and Wild, 2003; Miller, 1984; Box, 1971). The same bias applies to circular-linear regression.

A second option is a simulation approach from a parametric bivariate distribution $H(x, y)$, for example, a combination of a von Mises distribution for the circular and a Gaussian distribution for the linear data resulting in a joint distribution on the surface of the cylinder $[0, 2\pi) \times \mathrm{R}$. This last approach is not very flexible since it is limited to a combination of a few classical circular and linear distributions.

To model circular-linear dependencies a more modern, general and flexible approach consists of copula theory tools.

4. Bernstein copula approach of circular-linear dependence modeling

As important part of the methodology, in this section are exposed the benefits, presented the mathematical foundation, developed the numerical model, given the simulation algorithm and explained the computational model of Bernstein copula theory in the context of circular-linear data, such as fracture direction and length.

4.1. Conceptual model

From a statistical viewpoint, fracture strike data is usually modeled as a circular random variable, while fracture length is represented as a linear random variable. Thus, the strike-length relationship is modeled as a circular-linear probability distribution function.

For this kind of data, the periodicity condition of the directional variable must be included in the joint distribution function. For example, sometimes it is desirable to jointly model fracture direction and fracture length; in such case, the periodic condition must be added to the random vector modeling.

An alternative to direct bivariate $H(x, y)$ modeling is to generate the circular-linear distribution function through copula theory. Copula modeling is a very general and flexible approach to build multivariate distributions, since copulas are functions that capture the underlying dependence structure among variables.

Sklar's Theorem separates each univariate marginal distribution and captures dependence structure in the copula. Thus, copula theory fits the common two-step practice: to analyze and model each component (random variable) of the random vector individually, and, then proceed with a joint analysis.

It is usually found that dependence structure of fracture direction-length is complex, and can even depend on the resolution scale. Therefore, it is needed a procedure to model a wide spectrum of dependencies among random variables, not only the ones that are captured by correlation coefficients (Pearson, Spearman, Kendall). From approximation theory, Bernstein polynomials are a nonparametric approach that naturally satisfy the properties of distribution functions and copulas.

With copulas, simulation is also a two-step procedure. First, uniform random variables with the dependence structure are simulated, then the objective random variables are obtained by the quantile function of the univariate random variables modeled.

In summary, when there is dependence, the conceptual model considers the following:

- periodicity in the univariate (marginal) distribution function of the periodic variable,
- nonparametric marginal distributions,
- reduce or avoid transformation bias,
- can separate marginals modeling from the dependence structure,
- a nonparametric approach to model the dependence structure,
- periodicity in the copula density with respect to the directional variable.

4.2. Mathematical model

Copulas are functions $C : [0, 1]^2 \rightarrow [0, 1]$ with distribution function properties (Nelsen, 2006, p. 10) that became so popular because of the Sklar's Theorem (1959): Let H be a joint distribution function with continuous univariate margins F and G. Then there exists a unique bivariate copula C such that for all x, y in $\mathbb{R}$,

$$H(x, y) = C(F(x), G(y)) \tag{2}$$

One example of a copula is the product (or independent) copula $C_{\Pi}(u, v) = uv$ in which the corresponding univariate random variables U and V are independent.

Sklar's theorem allows to model H in two independent steps. On one side, the univariate marginal distributions F and G can be modeled one-by-one. On the other side, the dependence structure can be modeled by a copula C.

Inspired on the univariate empirical distribution function (Billingsley,

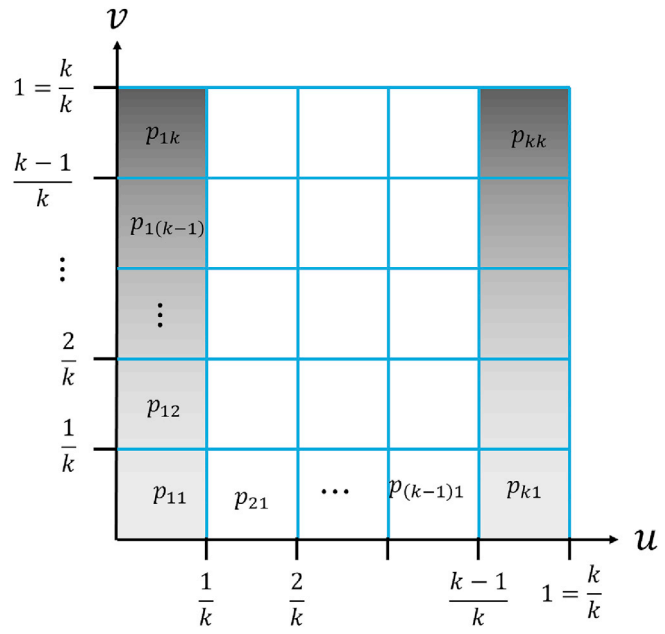

Fig. 1. Schematic representation of weights in which a discrete version of the copula periodicity condition (Equation (7)) is seen in dark.

1995, p. 268), one data-driven nonparametric approach would be to model C as the empirical copula (Deheuvels, 1979),

$$\widehat{C_n}\left(\frac{i}{n},\frac{j}{n}\right)=\frac{1}{n}\sum_{k=1}^{n}\mathbb{1}(rank(x_k)\le i, rank(y_k)\le j) \tag{3}$$

where n is the number of observations (x_k, y_k) of a random vector (X, Y), $i, j \in \{0, 1, \ldots, n\}$, and $\mathbb{1}$ is the indicator function (Erdely and Díaz-Viera, 2012).

One drawback of the empirical copula is the lack of continuity, but one important advantage is that it is nonparametric. A short survey on nonparametric copulas can be found on Joe (2014, sec. 5.10.3).

A nonparametric continuous version of the empirical copula can be obtained if it is approximated by means of Bernstein polynomials. The empirical Bernstein copula (Sancetta and Satchell, 2004) is defined as the Bézier surface in the form of a rectangular tensor product (Goldman, 2002, sec. 5.8.1)

$$C(u,v)=\sum_{i=1}^{n}\sum_{j=1}^{n}\widehat{C_n}\left(\frac{i}{n},\frac{j}{n}\right)\binom{n}{i}u^i(1-u)^{n-i}\binom{n}{j}v^j(1-v)^{n-j} \tag{4}$$

for every (u, v) in the unit square $I^2 := [0, 1] \times [0, 1]$, and where $\widehat{C_n}$ is defined in Equation (3).

Sometimes the probability density function f is preferred instead of the distribution function F. For example, for likelihood inference or for constraints on the random variable density. One constraint on f can be the periodicity for circular data (see Equation (1)).

In copula theory, given that C is an absolutely continuous copula, the bivariate density function (Casella and Berger, 2002, sec. 4.1) is

$$f_{X,Y}(x,y)=\frac{\partial^2 F}{\partial x\partial y}=c(F_X(x),F_Y(y))f_X(x)f_Y(y) \tag{5}$$

where $f_X(\cdot)$ and $f_Y(\cdot)$ represent the continuous marginal density functions and $c(\cdot, \cdot)$ (lower case) is the density function of the copula (also named copula density),

$$c(u,v)=\frac{\partial^2 C}{\partial u\partial v} \tag{6}$$

for $u := F_X(x)$, $v := F_Y(y)$.

Constraints on the univariate margins translate to constraints on the multivariate distribution functions and therefore to the copula. For example, in modeling periodic-linear random vectors such as wind direction-velocity or fracture direction-length, the periodicity condition on the directional variable, u, gives (Carnicero et al., 2013)

$$c(0,v)=c(1,v) \tag{7}$$

for $v \in [0, 1]$.

Some models satisfying this condition have been proposed, for example (Johnson and Wehrly, 1978), $c(u, v) = 2\pi h(2\pi(u+v))$ where $h(\cdot)$ is a univariate circular density. Another example is one of Carnicero et al. (2013).

Notice also that the independent copula density satisfies the periodicity requirements since $c_\Pi(u, v) = 1$. From this last result, independence for circular-linear variables will be tested as a distance to C_Π (Genest et al., 2006, 2007; Genest and Rémillard, 2004). Such independence test is available in the function indepTest of the copula R package (Hofert et al., 2015; Marius Hofert and Martin Machler, 2011; Ivan Kojadinovic and Jun Yan, 2010; Jun Yan, 2007).

4.3. Numerical model

The next discussion explains ideas from Carnicero et al. (2013) regarding copulas for directional data. From equations (4) and (6) the empirical Bernstein copula density is

$$c_B(u,v)=\sum_{i=0}^{n}\sum_{j=0}^{n}p_{ij}\beta(u|i,n-i+1)\beta(v|j,n-j+1) \tag{8}$$

where $\beta(x|a, b)$ is the beta density function usually found in statistics, and efficiently implemented in many software,

$$\beta(x|a,b)=\frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1} \tag{9}$$

where $B(a, b) = (a-1)!(b-1)!/(a+b-1)!$ and the weights are

$$p_{ij}=\frac{1}{n}\sum_{k=1}^{n}\mathbb{1}\left(\frac{i-1}{k}<u_k\le\frac{i}{k},\frac{j-1}{k}<v_k\le\frac{j}{k}\right) \tag{10}$$

for $i, j = 1, \ldots, n$.

These weights p_{ij}, the coefficients of the polynomial function, are equivalent to a two-dimensional histogram estimator in the domain I^2 of the copula (for details of the histogram estimator see Scott, 1992), i.e., if the pseudo-observations are defined as $u_i := F_X(x_i)$ and $v_i = F_Y(y_i)$, p_{ij} is $1/n$ times the number of points (u_i, v_i) inside the rectangle $(i-1, i]/k \times (j-1, j]/k$.

Once the copula density is available, if it is set that u corresponds to the periodic variable, the periodic constraint (7) on the copula density is satisfied when

$$p_{1j}=p_{kj} \tag{11}$$

For $j = 1, \ldots, k$, which will not in general be true. Carnicero et al. (2013) suggest the following correction

$$p_{1j}=p_{kj}=\frac{p_{1j}+p_{kj}}{2},\qquad j=1,\ldots,k \tag{12}$$

Fig. 1 illustrates the condition in (12) as shadowed bands, leaving the white cells untouched. Such condition ensures a strictly continuous, circular-linear estimated copula density.

With these weights can be computed a new empirical copula whose "empirical copula density" is periodic. This new empirical copula can then be used in (4) to approximate the copula.

Table 1
Parameters of the mixture of two von Mises distributions from which 400 directions where sampled.

family	proportion	$\mu(^\circ)$	κ
f0	0.3	0	10
f90	0.7	90	10

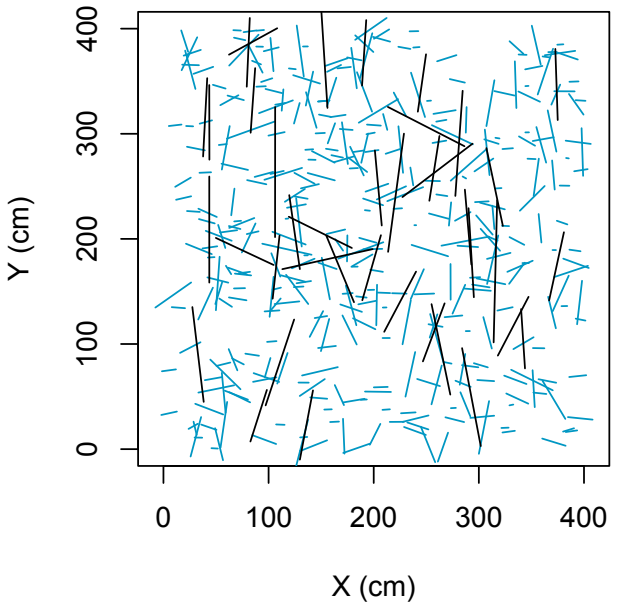

Fig. 2. Discrete Fracture Network graphical representation of the (synthetic) dataset. The 10% largest fractures are highlighted in black.

4.4. Simulation

Often, uncertainty assessment is made by means statistically equivalent simulations. To simulate replications from the random vector (X,Y) with the dependence structure inferred from the observed data, we use a particular case of the conditional distribution method (Nelsen, 2006, sec. 2.9; Erdely and Díaz-Viera, 2012, sec. 2) in which each of the marginals X and Y are a modeled with a nonparametric approach (Muñoz Pérez and Fernández-Palacín, 1987):

1. Generate two independent and continuous Uniform $(0,1)$ random variates u and t.
2. Set $v = c_u^{-1}(t)$. Where c_u^{-1} is the inverse of $c_u(v) = t = \frac{\partial C(u,v)}{\partial u}$.
3. If we estimate the quantile functions, Q and R, of X and Y by means of Bernstein-Kantorovich polynomials (Muñoz Pérez and Fernández-Palacín, 1987), the simulated pair is $(x, y) = (Q(u), R(v))$.

Being a polynomial, $c_u(v)$ is easily computed from (4). In contrast, the inverse $c_u^{-1}(t)$ have no analytic form, and must be solved numerically. One approach can be to find the root v of $G(v) := t - c_u(v)$. The function uniroot from the stats package finds such v.

4.5. Computational model

The methods in this article have been developed into a computer code. The software used is R because it is a free multi-platform environment for statistical computing and graphics, and is probably the most widely used among statisticians and data miners. An R package (available on request) has been developed for the results to be reproduced.

The periodic condition (11) is added to the copula matrix by taking the average of the first and the last column (12), and such average is assigned to each one of those columns.

Although De Casteljau's algorithm is fast and stable for computing multivariate Bernstein polynomials (Mainar and Peña, 2006), using the tensor representation of a polynomial is closely related to applying the de Casteljau's algorithm (Speleers, 2011). The tensor multiplication approach has been implemented in this package since it is easy to implement as a matrix outer multiplication.

5. Example: analysis, modeling and simulation of a synthetic dataset

5.1. The synthetic dataset

Although the methodology is valid for any fracture system, in this case it is studied a particular fracture network pattern broadly found in the Earth crust. Because of lack of real fracture data, a synthetic dataset has been generated according to geological knowledge:

> *Extension fracture*: Fracture formed by extension perpendicular to the fracture walls. The amount of extension can be minute, as for *joints*, or can be larger, as for *veins*
>
> Fossen (2010).

> Shear joints commonly occur in *conjugate sets* … shear joints occur in oblique conjugate sets whereas extension joints occur as longitudinal and transverse joints forming an orthogonal pair
>
> Singhal and Gupta (2010).

Further, some conjugate fracture systems have specific direction-length relationships:

> Miners refer to these two distinctive fracture directions as *coal cleats*: a *face cleat* composed of long continuous *dominant* fractures (systematic joints) that extend for many meters horizontally; and a *butt cleat* composed of shorter, somewhat more poorly developed fractures that terminate at right angles against the face cleat (cross joints).
>
> Davis et al. (2011).

From this structural geology knowledge, the fracture system synthetic dataset must satisfy:

- have two families f0, f90 (angle distribution is bimodal) with f0 being the dominant family,
- f90 with more fractures than f0,
- f90 fracture lengths shorter than f0.

We will refer to the dataset as *the dataset*, or *synthetic dataset*, and will be considered as if it were *real*. The 400-fracture dataset is composed of fracture centers $\overrightarrow{x}$ lying inside the square given by the coordinates $(0,0)$ and $(400,400)$, and simulated from a homogeneous Poisson process of intensity $= 1.0$; fracture strikes sampled from a mixture of two von Mises distributions (Table 1), angles varying in $[0,180)$ in a geographical coordinate system (North $= 0^\circ$, East $= 90^\circ$); and fracture lengths simulated from a lognormal distribution. Units of length correspond to the last property but, without loss of generality, centimeters (cm) will be used for fracture length. This length unit selection was motivated by the geological background from the previous section (section 5.1) but any other unit could have been chosen since it is known the scale invariance

Table 2
Circular statistics of the synthetic dataset directions.

Statistics	value
Median	88.458°
μ	89.728°
Standard deviation	44.063
Skewness	0.045

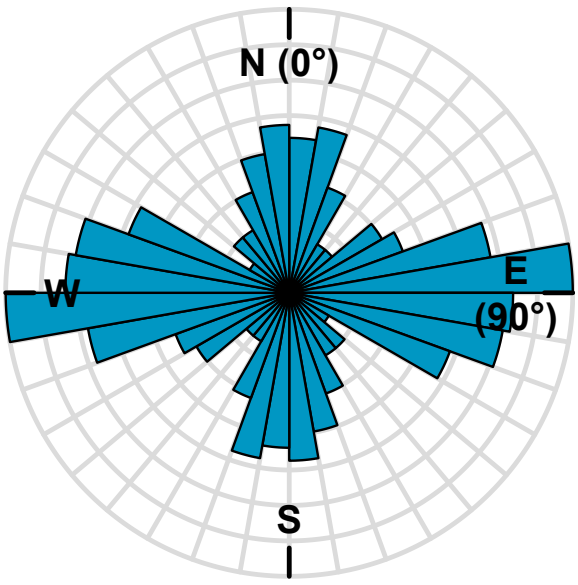

Fig. 3. Rose diagram of the dataset to be analyzed in which f0 is the N-S direction and f90 is in the E-W direction.

Table 3
Sample statistics of the synthetic dataset lengths.

Statistics	value
Median	20.389 cm
Mean	26.186 cm
Standard deviation	20.894 cm
Skewness	1.914

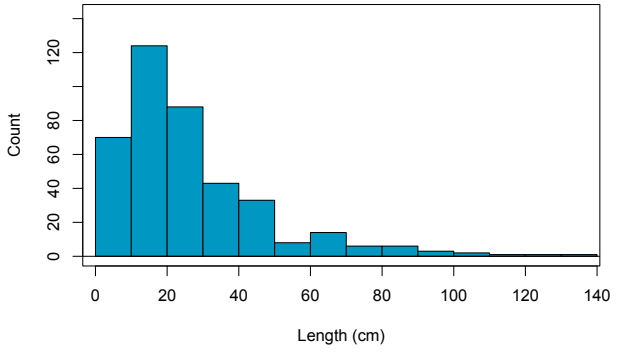

Fig. 4. Histogram of the synthetic dataset fracture lengths. Length ~ lognorm(meanlog = 3.00, sdlog = 0.73).

characteristic of many fracture networks. As a result, the DFN to be analyzed and modeled with the joint methodology is shown in Fig. 2. In order to analyze the dependence relationship between direction and length, fractures larger than 51 cm have been colored in black while the lower fractures are in light blue. This cutoff was based on the 90th percentile of the fracture length model, that is, about 10% percent of fractures are black.

Directions were generated using the function rmixedvonmises of the circular-R package (Agostinelli and Lund, 2013) with the parameters of Table 1. Some of its circular statistics (computed with the same package) are shown in Table 2. The circular mean (89.7°) is very similar to the circular median (88.5°). Both statistics points very close to the family f90. The very low skewness reveals that the distribution is highly symmetrical. These results can also be observed in Fig. 3 which also shows that the family f90 has a larger population than the horizontal family, as required by the model.

Lengths were generated using the function rlnorm of the stats-R package. The parameters of the lognormal distribution are *meanlog* = 3, *sdlog* = 0.73, the mean and standard deviation of the distribution on the log scale. The statistics in Table 3 show that the fracture mean and median are close to the greater population, and the high positive skewness value indicates little large fractures compared to small fractures. The histogram Fig. 4 also shows a tiny second mode at about 70 cm.

Fig. 5 shows the empirical cumulative distribution function of both direction and length. Although those data have very different behavior, the Bernstein-Kantorovich quantile function has a tight fit. For the direction, in this work it is preferred this flexibility over the periodic constraint in parametric models like von Mises.

For the bivariate exploratory data analysis, it is shown in Fig. 6 the direction-length scatterplot and its corresponding pseudo-observations scatterplot (also, psobs or *uv* plot). One difference of these two plots is that the former lies in the rectangle made up of the Cartesian product of the range of the two random variables while the latter is a plot limited to the square $[0,1] \times [0,1]$.

Since the analysis of the *uv* plot is not standard, let's explain how it is used. Values in the axes of the pseudo-observations plot (Fig. 6) corresponds to probabilities of quantiles. For instance, in the horizontal axis, the (median) angle 90° corresponds to $u = 0.5$ while the 3rd quartile corresponds to $u = 0.75$, and the angle 0° (or 180°) to the value 0 (or 1 because of periodicity). Similarly for the vertical axis (lengths).

The psobs plot allows a visual overview of the dependence structure. This way, it can be studied the effect of the pseudo-observations plot, or equivalently the dependence structure, on the true scatterplot. The rank correlation coefficient $\rho_M = 0.635$, computed with the codes provided in Tu (2015), suggests that the dataset is not independent.

A quantitative copula-based independent test can be carried out with the function indepTest from the copula package. The synthetic dataset gives a test statistic of 0.408 and a p-value of 5×10^{-4}, which means that independence is rejected, so that it is worth to continue to model the dependence structure.

Notice in the scatterplot that the main mean directions (0° and 90°)

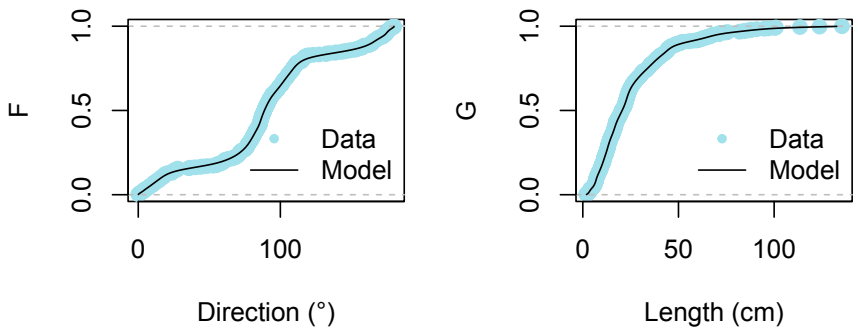

Fig. 5. Empirical cumulative distribution function and its Bernstein-Kantorovich model for the synthetic data. Left: Direction, right: Length.

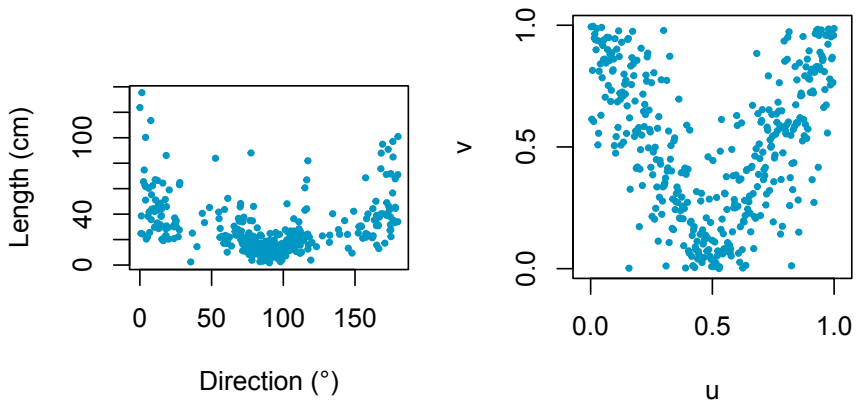

Fig. 6. Synthetic dataset Scatterplot of length vs angle (left), and its corresponding direction-length pseudo-observations plot (right).

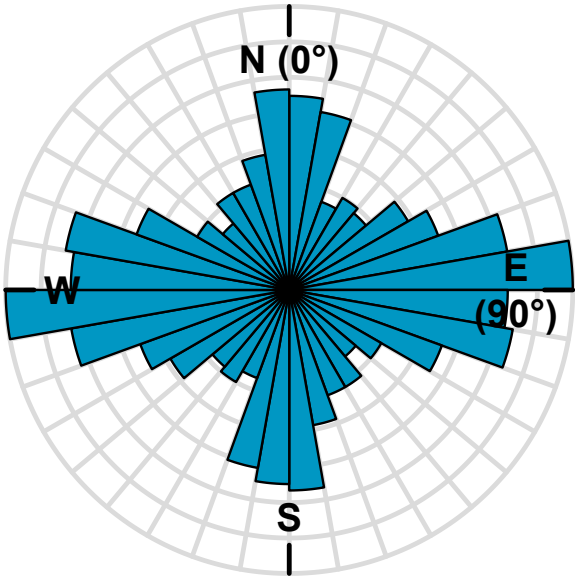

Fig. 7. Rose diagram of one simulation of the dataset. Compare with Fig. 3.

Table 4
Circular statistics of directions from one simulation of the dataset. Compare with Table 2.

Statistics	value
Median	83.930°
μ	85.431°
Standard deviation	48.920
Skewness	0.166

show two modes, but in contrast to the rose diagram which shows the same modes values, the higher mode corresponds to family f0. This family does not allow fractures shorter than 18 cm. In contrast, near 90°, there exits only small fractures and no large fracture is statistically allowed. This phenomenon is due to the dependence structure shown in the pseudo-observations plot. Notice also that there exist more small fractures (say, below 51 cm) than large fractures all along interval $[0, 180)$.

5.2. The standard methodology

It is advised the distinction between the words synthetic and simulated, we will use the former for the data analyzed and the later for the data obtained by the methodology shown in this work. The data can only be one dataset while there could be multiple simulations for a given dataset. Following, it is shown the results of one simulation.

In this section are shown the results when using the standard methodology in which directions and angles are simulated independently using the models of Fig. 6. For better comparison, the bin widths and locations in the rose diagram of the synthetic (Fig. 3) and simulated dataset (Fig. 7) are the same. For the bins in the histograms of lengths applies an analogue case. Also to the plotting range of the scatterplots.

The rose diagram of the independent simulation (Fig. 7) looks with less contrast in the modes and valleys, i. e., it seems that the model tries to smooth the behavior. Different from this, the modes and valleys locations are satisfactorily reproduced. Even some subtle skewed parts are reproduced. Its circular statistics (Table 4) reflects quantitatively that the model has a good fit; however, the standard deviation seems to reinforce the suspicion of the smoothing effect.

The independently simulated lengths were also reproduced. Notice for instance that the valley at about 55 cm is reproduced in the histogram (Fig. 8). The subtle mode at about 70 cm is showing too. On the quantitative side (Table 5), similar to the direction, the standard deviation is larger than in the synthetic dataset. The median and the mean different because of skewness as in the synthetic dataset but close to their corresponding model values.

This agreement between the simulation and the dataset is also observed in their closeness to the models (Fig. 9). This is the result of the non-parametric property of the Bernstein-Kantorovich Quantile function.

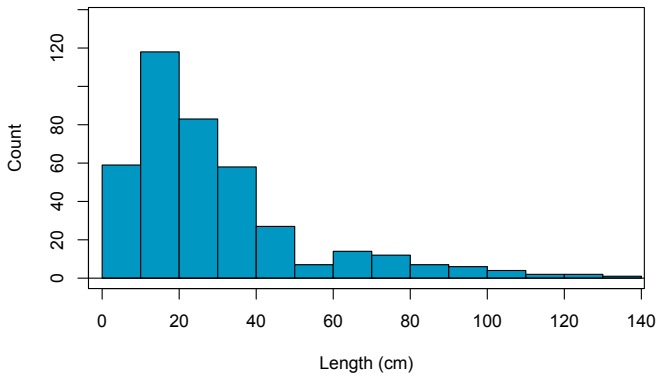

Fig. 8. Histogram of lengths from one simulation of the dataset. Compare with Fig. 4.

Table 5
Sample statistics of lengths from one simulation of the dataset. Compare with Table 3.

Statistics	value
Median	22.877 cm
Mean	29.400 cm
Standard deviation	23.930 cm
Skewness	1.752

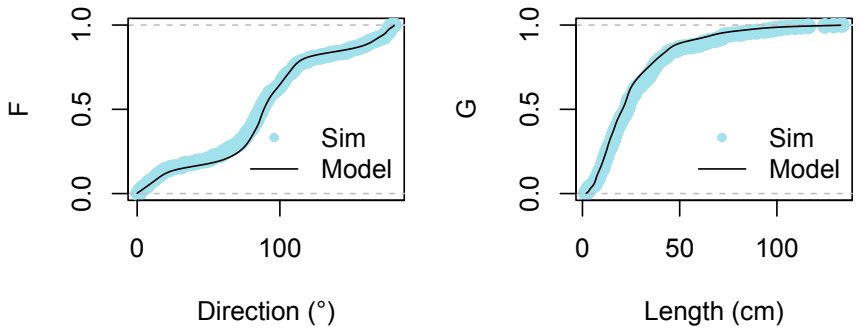

Fig. 9. Empirical cumulative distribution function for the simulated data and the Bernstein-Kantorovich model of Fig. 5. Left: Direction, right: Length.

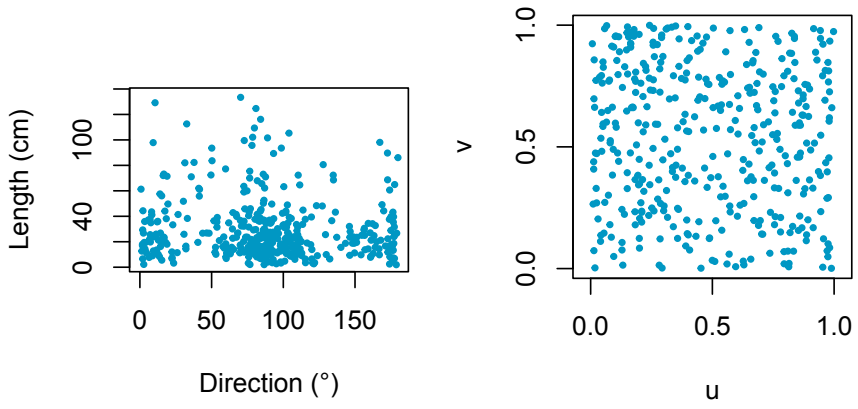

Fig. 10. Scatterplot and pseudo-observations plot of the dataset simulated with the standard methodology. Compare with Fig. 6.

Notice, however, that in the bivariate case (Fig. 10) the modeling were not as satisfactorily as in the univariate case. The points in the scatterplot and in the pseudo-observations plot are more disperse. The correlation coefficient is also clearly different ($\rho_M = 0.016$). Neither the scatterplot nor the *uv* plot show the bivariate structure. The reason: it was not considered in the simulation. The standard methodology produces results like this one, and the pseudo-observations plot will tend to be uniform as in the right of Fig. 10.

The resulting DFN (Fig. 11) pays the price by showing the 10% largest fractures not as vertically as observed in Fig. 2, but, to the contrary, now there seems to be more large fractures in the E-W direction than in the vertical direction.

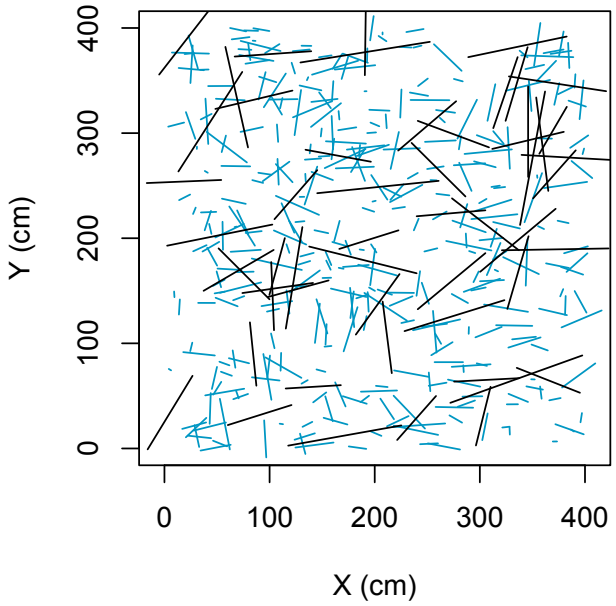

Fig. 11. DFN of one independent simulation of the dataset. Compare with Fig. 2.

5.3. *The Bernstein copula approach*

The methodology in this work does consider the bivariate structure through the Bernstein copula of the pseudo-observations in Fig. 6. With this copula model, new simulated pseudo-observations (Fig. 12, left) were sampled using steps 1 and 2 of the simulation algorithm section 4.4. Then the simulated direction-length values (Fig. 12, right) were obtained by the step 3 in which the quantile functions are the ones shown in Fig. 6.

The power of copula theory allowed to have the same marginals values as in the independent case, i. e., the rose diagram, the length histogram and the statistics are exactly the same as in the independent case section. The difference is in the dependence structure. Fig. 12 shows the scatterplots. Notice the contrast with the independent case and the agreement with the synthetic dataset. Quantitatively, the rank correlation coefficient ($\rho_M = 0.571$) also agrees with the dataset.

The corresponding DFN shows, as in the synthetic dataset, that there is a preference of the largest fractures for the vertical direction. Nevertheless, there is the possibility of getting non-vertical large fractures.

6. Results and discussion

Although by construction, the univariate nonparametric approach to model the marginal distribution of direction is not periodic, it mimics pretty well the periodic behavior of fracture directions (See Fig. 7 and Table 4), and even shows the skewed parts in the rose diagram. For lengths, it is also observed a coherent match between data and simulation (See Fig. 8 and Table 5).

On the bivariate case, in Fig. 12 it is shown that the Bernstein copula approach reproduces tightly the dependence structure of direction and length: small fractures correspond to family f90 ($U_{angle} = 0.5$) and large

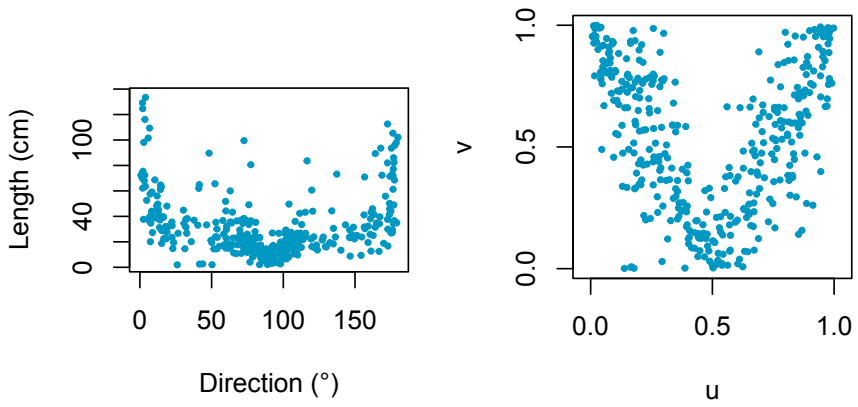

Fig. 12. Scatterplot and pseudo-observations plot of the dataset simulated with the copula methodology. Compare with Fig. 6 and with Fig. 10.

fractures to f0. The method also shows a very good agreement even in the subtle characteristic that there is almost no probability of getting very small fractures for the angle 0°, ($U_{angle} = 0$ and $V_{length} < 0.2$).

On the contrary, the standard methodology produces results totally different. In this example there is a larger population of the largest fractures in the E-W direction than in the N-S direction. Further, very small fractures appears in the N-S direction, but in synthetic dataset there are no fractures of length less than 18 cm.

Notice also that there is also a good correspondence of the DFN (Fig. 2) and the simulated DFN (Fig. 13) that takes into account the dependence structure. In general, all the variables involved are well reproduced, both graphically and in its descriptive statistics, and both marginally and jointly.

Unforeseen, it can be observed in the rose diagram (Fig. 7) and in the scatterplot (Fig. 12) that angles are apparently a little more diffused in simulations than in the data. This effect could be due to Bernstein polynomials which have a smoothing effect or it could be just an effect of this unique simulation, other simulations could not show such effect.

Because fracture size and directions are very important in reservoir fluid flow paths, and to estimate percolation properties of a given fracture system, this methodology is much better than omitting such dependencies.

Although there exists a multivariate approach for Bernstein copulas (Wei and Scheffer, 2012), the bivariate case in this work is restrictive in the number of variables: two. Another case worth considering is the spatial dependence of intensity on fracture properties.

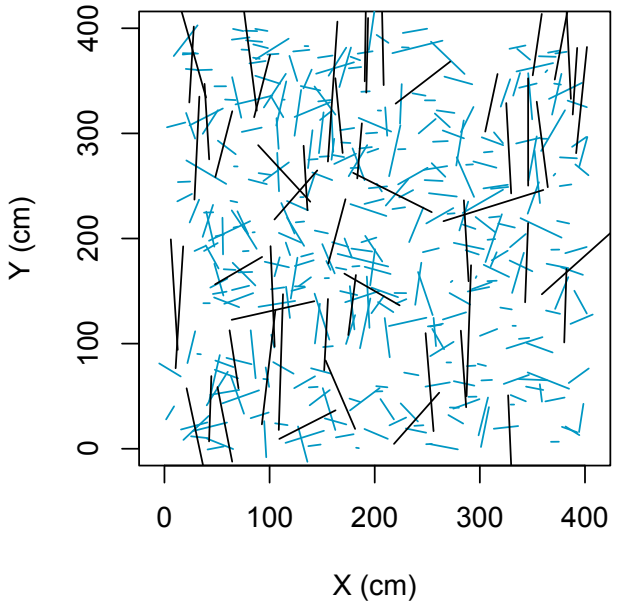

Fig. 13. DFN of one Bernstein copula-based simulation of the dataset. Compare with Fig. 2 and with Fig. 13.

7. Conclusions and future work

In contrast to the restrictive assumption of linear or log-linear relationship between variables, the Bernstein copula approach allows to statistically investigate complex dependence structures driven by observed data (direction-length in this work) in a very flexible fashion. In particular, the periodic condition of Carnicero et al. (2013) extends the approach to include variables such as fracture direction.

Using a copula approach for the dependence structure allows to avoid the bias produced when transforming random variables involved in simulations, for example log-transforming length.

The nonparametric approaches used for the marginal distributions and the copula allowed a very good match of a simulation of a DFN even in the presence of distributional skewness. This will allow to estimate, through simulation analysis, more realistic percolation properties of fracture systems. Another advantage with this nonparametric approach is the ease of use since no maximum likelihood or goodness-of-fit test is required.

With the methodology of this work, the only dependence missing to model is the relationship between spatial location and fracture properties (direction-length).

Immediate future work could be executed on the simplest multivariate case: three-variate case in which two of them are independent. For example, modeling direction-length-aperture, and allowing direction-aperture be independent.

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