A consistent first-order model for relativistic heat flow

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Abstract
This paper revisits the problem of heat conduction in relativistic fluids, associated with issues concerning both stability and causality. It has long been known that the problem requires information involving second-order deviations from thermal equilibrium. Basically, any consistent first-order theory needs to remain cognizant of its higher order origins. We demonstrate this by carrying out the required first-order reduction of a recent variational model. We provide an analysis of the dynamics of the system, obtaining the conditions that must be satisfied in order to avoid instabilities and acausal signal propagation. The results demonstrate, beyond any reasonable doubt, that the model has all the features one would expect of a real physical system. In particular, we highlight the presence of a second sound for heat in the appropriate limit. We also make contact with previous work on the problem by showing how the various constraints on our system agree with previously established results.

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1. Context

Relativistic thermodynamics continues to provide interesting challenges, in particular in the context of dissipative and nonlinear phenomena. The issues involved range from direct applications in various areas of physics to fundamental problems like the nature of time (vis-à-vis the second law of thermodynamics) and the formation of structures at nonlinear deviations from thermal equilibrium. Much recent work has been motivated by the modelling of complex astrophysical systems, like neutron stars [1], and cosmology [2]. There has also been a resurgence of interest in dissipative systems in the context of colliders like the RHIC at Brookhaven and the LHC at CERN [3–5]. These latter developments, which have to a large extent been driven by the need to understand the dynamics of a hot quark–gluon plasma, are often linked with underlying principles like the AdS/CFT conjecture and holography [6]. Even though the problem dates back to the origins of relativity theory, it remains (in a slightly different guise) at the forefront of modern thinking.

According to the established consensus view, one must account for second-order deviations from thermal equilibrium in order to achieve causality and stability. This is certainly the lesson from the celebrated work of Israel and Stewart [7, 8], see [9–11] for recent work on the problem. We have recently revisited the key points in the context of heat conduction [12], taking a multi-fluid prescription based on Carter's convective variational formulation for relativistic fluids [13] as our starting point. This is a mathematically elegant approach that has the flexibility required to account for the physics that we need to consider. A particularly appealing feature of the variational approach is that, once an 'equation of state' for matter is provided, the theory provides the relation between the various currents and their conjugate momenta [1]. The variational analysis leads to a second-order model which has the key elements required for causality and stability; in particular, it clarifies the role of the inertia of heat (e.g., the effective mass associated with phonons). This effect enters the model in an intuitive fashion in terms of entrainment between the matter and heat [14]. As demonstrated by Priou [15] some time ago, the final variational model is formally equivalent to the Israel–Stewart construction. This exercise demonstrates clearly that the relaxation associated with causal heat transport is determined by the thermal inertia. At the end of the day, the theoretical framework becomes rather intuitive and the physics involved seems natural.

Does this mean that no troublesome issues remain in this problem area? Not quite. First of all, it is clear that the need to introduce additional parameters (e.g., the relevant relaxation times) and keep track of higher order terms (fluxes of the fluxes etcetera) make actual applications rather complex. Secondly, we are not much closer to considering systems that deviate significantly from equilibrium, such that there is no natural 'small' parameter to expand in. The variational model sheds some light on this regime by clarifying the role of the temperature in systems out of equilibrium, but there is some way to go before we understand issues associated with, for example, any 'principle of extremal entropy production' and instabilities that lead to structure formation. Finally, despite the obvious successes of the extended thermodynamics framework [16], there is no universal agreement concerning the validity (and usefulness) of the results. To some extent this is natural given the interdisciplinary nature of the problem; to make progress we need to account for both thermodynamical principles and fundamental general relativity. This leads to a range of deep questions concerning, in particular, the actual meaning of the variables involved in the different models (e.g., the entropy). The ultimate theory must have a clear link with statistical physics and even information theory. Our efforts are not yet at that level. Basically, we need to continue to make progress if we want to address fundamental problems in, for example, cosmology.

This paper sets a rather more modest target; we want to explore the extent to which a 'first-order' formulation for heat conduction in general relativity is viable. The question may seem somewhat odd given that we have already acknowledged the need to account for (at least) second-order contributions. However, it is interesting to ask whether a first-order model may nevertheless be useful (possibly in a somewhat restricted sense). We will demonstrate that this is, indeed, the case. Noting that the original first-order models, due to Eckart [17] and Landau and Lifshitz [18], were incomplete we develop a consistent framework that includes the key thermal relaxation. We then consider the properties of this model and show that it can be made both stable and causal (making contact with the classic work by Hiscock and Lindblom [19, 20]). This does not mean that the system may not exhibit instabilities. On the contrary, instabilities are in a sense generic in these problems [21], but the analysis sheds further light on the nature of these instabilities and also elucidates the stabilizing role of the thermal inertia. Moreover, the discussion provides insight into the emergence of second sound (an effect that has been experimentally verified in low-temperature crystals) associated with the

heat transport. This provides a key link to systems that exhibit superfluidity and demonstrates the potential for a unified treatment of heat transport in normal and superfluid matter.

2. Thermal dynamics

We take as our starting point our recent variational analysis of the relativistic heat problem [12]. The model is phenomenological and assumes that the entropy component can be treated as a 'fluid'. In essence, this implies that the mean free path of the phonons is taken to be small compared to the model scale. We then consider two fluxes, one corresponding to the matter flow and one which is associated with the entropy. The latter is essentially treated as a massless (zero rest-mass) fluid. The dynamics then follows from a Lagrangian which depends on the relative flow of the two fluxes. The associated entrainment turns out to be a crucial feature of the problem [12, 14, 22].

We assume that the particle number is large enough that the fluid approximation applies and there is a well-defined matter current, n^a. Moreover, we adopt the multi-fluid view and treat the entropy as an effective fluid with flux s^a. This current is in general not aligned with the particle flux. The misalignment is associated with the heat flux and leads to entropy production.

As in the case of a generic two-fluid system (see [23] for an example in the case of a cool relativistic superfluid), the starting point is the definition of a relativistic invariant Lagrangian Λ. Assuming that the system is isotropic, we take Λ to be a function of the different scalars that can be formed by the two fluxes[1]. From n^a and s^a we can form three scalars:

$$n^2 = -n_a n^a, \qquad s^2 = -s_a s^a, \qquad j^2 = -n_a s^a. \tag{1}$$

An unconstrained variation of Λ then leads to

$$\delta\Lambda = \frac{\partial\Lambda}{\partial n}\delta n + \frac{\partial\Lambda}{\partial s}\delta s + \frac{\partial\Lambda}{\partial j}\delta j. \tag{2}$$

Changing the passive density variations for dynamical variations of the worldlines generated by the fluxes and the metric (as discussed in [1]) we find that

$$\delta\Lambda = \left[-2\frac{\partial\Lambda}{\partial n^2}n_a - \frac{\partial\Lambda}{\partial j^2}s_a\right]\delta n^a + \left[-2\frac{\partial\Lambda}{\partial s^2}s_a - \frac{\partial\Lambda}{\partial j^2}n_a\right]\delta s^a$$
$$+ \left[-\frac{\partial\Lambda}{\partial n^2}n^a n^b - \frac{\partial\Lambda}{\partial s^2}s^a s^b - \frac{\partial\Lambda}{\partial j^2}n^a s^b\right]\delta g_{ab}. \tag{3}$$

From this result we can read off the conjugate momentum associated with each of the fluxes,

$$\mu_a = \frac{\partial\Lambda}{\partial n^a} = g_{ab}(\mathcal{B}^{\mathrm{n}}n^b + \mathcal{A}^{\mathrm{ns}}s^b), \qquad \theta_a = \frac{\partial\Lambda}{\partial s^a} = g_{ab}(\mathcal{B}^{\mathrm{s}}s^b + \mathcal{A}^{\mathrm{ns}}n^b), \quad (4)$$

where we have introduced the coefficients [1, 12]

$$\mathcal{B}^{\mathrm{n}} \equiv -2\frac{\partial\Lambda}{\partial n^2}, \qquad \mathcal{B}^{\mathrm{s}} \equiv -2\frac{\partial\Lambda}{\partial s^2}, \qquad \mathcal{A}^{\mathrm{ns}} \equiv -\frac{\partial\Lambda}{\partial j^2}. \tag{5}$$

The energy–momentum tensor is obtained by noting that the displacements of the conserved currents induce a variation in the spacetime metric and therefore the variations of the fluxes, δn^a and δs^a, are constrained. The energy–momentum tensor is thus found to be

$$T_a{}^b = \mu_a n^b + \theta_a s^b + \Psi\delta_a{}^b, \tag{6}$$

where we have defined the generalized pressure, Ψ, as

$$\Psi = \Lambda - \mu_a n^a - \theta_a s^a. \tag{7}$$

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[1] The natural way to account for viscosity is to allow Λ to depend also on the associated stresses [24].

As a result of the coordinate invariance associated with general relativity, the divergence of the energy–momentum tensor (6) vanishes. For an isolated system, we can express this requirement as an equation of force balance

$$\nabla_b T_a{}^b = f_a^{\mathrm{n}} + f_a^{\mathrm{s}} = 0, \tag{8}$$

where the individual force densities are [12]

$$f_a^{\mathrm{n}} = 2n^b \nabla_{[b} \mu_{a]} + \mu_a \nabla_b n^b, \tag{9}$$

$$f_a^{\mathrm{s}} = 2s^b \nabla_{[b} \theta_{a]} + \theta_a \nabla_b s^b. \tag{10}$$

We note that in order to obtain the energy–momentum tensor (6) we needed to impose the conservation of the fluxes as constraints on the variation [1]. However, the equations of motion, (9) and (10), still allow for non-vanishing production terms. If we, for simplicity, consider a single particle species, the matter current is conserved and we have $\nabla_a n^a = 0$. This removes the second term from the right-hand side of (9). In contrast, the entropy flux is generally not conserved. In accordance with the second law, we must have

$$\nabla_a s^a = \Gamma_{\mathrm{s}} \geqslant 0. \tag{11}$$

2.1. Temperature

To make progress, we need to connect the general variational results with the relevant thermodynamical concepts. In doing this it makes sense to consider a specific choice of frame. In the context of a single (conserved) species of matter, we see that force f_a^{n} is orthogonal to the matter flux, n^a, and therefore it has only three degrees of freedom. Furthermore, because of the force balance (8), we also have $n^a f_a^{\mathrm{s}} = 0$. This suggests that it is natural to focus on observers associated with the matter frame. We therefore introduce the four-velocity u^a such that $n^a = nu^a$, where $u_a u^a = -1$ and n is the number density measured in this frame.

Having chosen to work in the matter frame (in the spirit of Eckart [17]), we can decompose the entropy current and its conjugate momentum into parallel and orthogonal components. The entropy flux is then expressed as

$$s^a = s^*(u^a + w^a), \tag{12}$$

where w_a is the relative velocity between the two fluid frames, and $u^a w_a = 0$. Letting $s^a = su_{\mathrm{s}}^a$, where u_{s}^a is the four-velocity associated with the entropy flux, we see that $s^* = s\gamma$, where γ is the redshift associated with the relative motion of the two frames[2]. This illustrates the subjective nature of entropy. It is an observer-dependent quantity, not an absolute notion.

Similarly, we can write the thermal momentum as

$$\theta_a = \theta^* u_a + \mathcal{B}^{\mathrm{s}} s^* w_a = \left(\mathcal{B}^{\mathrm{s}} s^* + \mathcal{A}^{\mathrm{ns}} n \right) u_a + \mathcal{B}^{\mathrm{s}} s^* w_a. \tag{13}$$

This leads to the temperature measured in the matter frame,

$$-u^a \theta_a = \theta^* = \mathcal{B}^{\mathrm{s}} s^* + \mathcal{A}^{\mathrm{ns}} n. \tag{14}$$

In essence, this quantity represents the effective mass of the entropy component. Returning to the stress–energy tensor and making use of the projection orthogonal to the matter flux, we find that the heat flux (energy flow relative to the matter) is given by

$$q_a = - \perp_{ab} u_c T^{bc} = s^* \theta^* w_a, \tag{15}$$

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[2] In the following, we will use asterisks to highlight when the entropy density and the temperature (quantities that would be naturally considered in the entropy frame) are, in fact, expressed in the matter frame. In the linearized model, this distinction is not necessary.

where we have used the projection

$$\perp^{ab} = g^{ab} + u^a u^b. \tag{16}$$

Defining the new variables $\tilde{\sigma}^a = s^* w^a$ and $\tilde{p}_a = \mathcal{B}^s s^* w_a$, the energy density measured in the matter frame can be obtained by a Legendre-type transform on Λ. That is, we have

$$\rho = u_a u_b T^{ab} = -\Lambda + \tilde{p}_a \tilde{\sigma}^a. \tag{17}$$

The relevance of the new variables becomes apparent if we consider the fact that the *dynamical* temperature in (14) agrees with the *thermodynamical* temperature that an observer moving with the matter would measure. In other words, we have

$$\theta^* = \left. \frac{\partial \rho}{\partial s^*} \right|_{n,\tilde{p}}, \tag{18}$$

where $\rho = \rho(n, s^*, \tilde{p})$. This is, of course, the standard definition of temperature as energy per degree of freedom of the system. Mathematically, the temperature is obtained from the variation of the energy with respect to the entropy in the observer's frame (keeping the other thermodynamic variables fixed).

This result is not trivial. The requirement that the two temperature measures agree determines the additional state parameter, $\tilde{p}$, to be held constant in the variation of ρ. The importance of the chosen state variables is emphasized further if we note that, when the system is out of equilibrium, the energy depends on the heat flux (encoded in $\tilde{\sigma}^a$ and $\tilde{p}_a$). This leads to an *extended* Gibbs relation (similar to that postulated in many approaches to extended thermodynamics [16]),

$$d\rho = \mu \, dn + \theta^* \, ds^* + \tilde{\sigma} \, d\tilde{p}. \tag{19}$$

This result arises naturally from the variational analysis.

According to the traditional view, thermodynamic properties like pressure and temperature are uniquely defined only in equilibrium. Intuitively, this makes sense since, in order to carry out a measurement, the measuring device must have time to reach 'equilibrium' with the system. A measurement is only meaningful as long as the timescale required to obtain a result is shorter than the evolution time for the system. However, this does not prevent a generalization of the various thermodynamic concepts (as described above). The procedure may not be 'unique', but one should at least require the generalized concepts to be internally consistent within the chosen extended thermodynamics model. Our model satisfies this criterion.

2.2. Causal heat flow

The variational model encodes the finite propagation speed for heat, as required by causality. To see this, we use the orthogonality of the entropy force density f_s^a with the matter flux, solve for the entropy production rate Γ_s and finally impose the second law of thermodynamics. It is natural to express the result in terms of the heat flux q^a, defined by

$$s^a = s^* u^a + \frac{1}{\theta^*} q^a. \tag{20}$$

We also let the conjugate momentum take the form

$$\theta_a = \theta^* u_a + \beta q_a, \tag{21}$$

where we have defined

$$\beta = \left(\frac{1}{s^*} - \frac{\mathcal{A}^{ns} n}{s^* \theta^*} \right). \tag{22}$$

With these definitions, we impose the second law of thermodynamics by demanding that the entropy production is a quadratic in the sources, i.e.

$$\Gamma_s = \frac{q^2}{\kappa \theta_*^2} \geqslant 0, \tag{23}$$

where $\kappa > 0$ is the thermal conductivity. This means that the heat flux is governed by

$$\tau \left(\dot{q}^a + q_c \nabla^a u^c \right) + q^a = -\tilde{\kappa} \perp^{ab} \left(\nabla_b \theta^* + \theta^* \dot{u}_b \right), \tag{24}$$

where $\dot{q}^a = u^b \nabla_b q^a$ and $\dot{u}^a$ is the four-acceleration (in the following, dots represent time derivatives in the matter frame). We have also introduced

$$\tilde{\kappa} \equiv \frac{\kappa}{1 + \kappa \dot{\beta}}, \tag{25}$$

while the thermal relaxation time is given by

$$\tau = \frac{\kappa \beta}{1 + \kappa \dot{\beta}}. \tag{26}$$

The final result (24) is the relativistic version of the so-called Cattaneo equation [12]. From the analysis we learn that the entropy entrainment, encoded in $\mathcal{A}^{ns}$, plays a key role in determining the thermal relaxation time τ. This agrees with the implications of extended thermodynamics and echoes recent results in the context of Newtonian gravity [14].

2.3. The matter flow

The heat problem has two dynamical degrees of freedom. So far, we have focused on the entropy. In addition to the relativistic Cattaneo equation (24), we have a momentum equation for the matter component. From (9) it follows that this equation can be written as

$$\mu \dot{u}_a + \perp_a^b \nabla_b \mu + \alpha \dot{q}_a + \dot{\alpha} q_a + \alpha q^b \nabla_a u_b = \frac{1}{n} f_a^n. \tag{27}$$

Here, we have represented the matter momentum by

$$\mu_a = \mu u_a + \alpha q_a, \tag{28}$$

where μ is the chemical potential (in the matter frame) and

$$\alpha = \frac{\mathcal{A}^{ns}}{\theta^*}. \tag{29}$$

This means that we have

$$\alpha = \frac{1 - \beta s^*}{n}. \tag{30}$$

Given these definitions, we have (cf (23))

$$-f_a^n = f_a^s = -\frac{1}{\tilde{\kappa}} \left[s^* - \frac{\beta q^2}{(\theta^*)^2} \right] q_a. \tag{31}$$

It is useful to note that this implies that the force has a term that is linear in q^a. This will be important later.

These relations complete our summary of the heat-conduction model developed in [12].

3. A consistent first-order model

The model we have described contains terms that enter at second order of deviation from thermal equilibrium, e.g., terms that are second order in the heat flux q^a. Moreover, it is clear that key effects (like the entropy entrainment) arise from the presence of second-order terms in the Lagrangian Λ. Having said that, it is obviously the case that we can truncate the model at first order. This does *not* take us back to the original first-order model discussed by Eckart [17]. Crucially, the thermal relaxation remains. Basically, this reflects the simple fact that you need to know the energy of a system to quadratic order in order to develop the complete linear equations of motion. Noting this, it is interesting to consider the features of this new first-order model. First of all, we can expect to get a clearer understanding of some of the general features of the variational model. Secondly, we may also find that this, much simpler, model is adequate for many situations of practical interest.

3.1. The linear model

We want to restrict our analysis to first-order deviations from equilibrium. Thermal equilibrium corresponds to $q^a = 0$, no heat flux, and $\dot{u}^a = 0$, no matter acceleration. Moreover, in the simplest cases there should be no shear, divergence or vorticity associated with the flow, i.e. we will have $\nabla_a u^a = 0$ and $\nabla_b u^a = 0$ as well[3]. Treating all these quantities as being of first order, and noting that

$$u_b \dot{q}^b = -q^b \dot{u}_b, \tag{32}$$

contributes at second order, we arrive at two momentum equations; from (27) we have

$$\mu \dot{u}_a + \perp^b_a \nabla_b \mu + \alpha \dot{q}_a + \left(\dot{\alpha} - \frac{s}{n\tilde{\kappa}} \right) q_a = 0, \tag{33}$$

while (24) leads to

$$\tau \dot{q}_a + q_a + \tilde{\kappa} \left(\perp^b_a \nabla_b T + T \dot{u}_a \right) = 0. \tag{34}$$

We also have the two conservation laws

$$\nabla_a n^a = 0, \tag{35}$$

$$\nabla_a s^a = 0. \tag{36}$$

In these equations, we have used the fact that s^* and θ^* differ from the equilibrium values s and T only at second order. Moreover, to first order the pressure p is obtained from the standard equilibrium Gibbs relation,

$$\nabla_a p = n\nabla_a \mu + s\nabla_a T. \tag{37}$$

Finally, we have the fundamental relation

$$\rho + p = \mu n + sT. \tag{38}$$

By comparing (33) and (34) to Eckart's results it becomes apparent to what extent the first-order model relies on its higher order origins. Specifically, α and (therefore) τ depend on $\mathcal{A}^{ns}$ and the entropy entrainment, cf (29). These effects rely on quadratic terms in the Lagrangian, and hence would not be present in a model that includes only first-order terms from the outset. Hence, they are absent in Eckart's model.

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[3] There are obviously many relevant problems that require a more 'complicated' equilibrium, e.g. rotating stars and an expanding universe in cosmology. It is, however, straightforward to extend our analysis to these cases.

In order to analyse the dynamics of the heat problem, we will consider perturbations (represented by δ) away from a uniform equilibrium state. Working in the matter frame, the equilibrium configuration is such that $u^a = [1, 0, 0, 0]$ and we have $q_a = \dot{u}_a = 0$. We can also ignore $\dot{\alpha}$ and $\dot{\beta}$, since the equilibrium configuration is uniform, which means that we can replace $\tilde{\kappa}$ by κ. This means that we are left with the two equations (since the perturbation commutes with the time derivative in this case):

$$\mu\delta\dot{u}_a + \perp_a^b \nabla_b\delta\mu + \alpha\delta\dot{q}_a - \frac{s}{n\kappa}\delta q_a = 0 \tag{39}$$

and

$$\tau\delta\dot{q}_a + \delta q_a + \kappa \perp_a^b \nabla_b\delta T + \kappa T\delta\dot{u}_a = 0. \tag{40}$$

It is worth noting that we can combine these two to obtain

$$(p + \rho)\,\delta\dot{u}_a + \perp_a^b \nabla_b\delta p + \delta\dot{q}_a = 0. \tag{41}$$

The last two equations ((40) and (41)) are, not surprisingly, identical to the first-order reduction of the Israel–Stewart model. This means that the problem is relatively well explored. In particular, the conditions required for stability and causality were derived by Hiscock and Lindblom [19, 20], see also Olson and Hiscock [25], quite some time ago. However, there are good reasons to revisit the problem. Most importantly, there is clear evidence from the recent literature (cf, discussions of the relevance of the thermal relaxation and the role of the coupling between the four-acceleration and the heat flux [26–28]) that the key lessons from almost three decades ago have not been appreciated. To some extent this could be due to the fact that the Hiscock–Lindblom analysis is rather involved. Our aim is to clarify the main issues in the simpler context of heat conduction (ignoring viscosity). We also want to emphasize aspects that were only mentioned in passing in early work. Particularly relevant in this respect is the existence of second sound; an effect that is prominent in superfluids but which has also been observed in low-temperature crystals. We will demonstrate how the second sound emerges within the causal heat-conduction model. The overarching aim is to establish, beyond any reasonable doubt, that the model represented by (40) and either (39) or (41) has all the properties expected of a reliable model for heat conduction in general relativity.

3.2. Transverse waves: stability

Working in the frame associated with the background flow, it is easy to show that all terms in (39) and (40) are orthogonal to the background four-velocity u^a. In other words, the equations only have spatial components. Given this, we may erect a local Cartesian coordinate system associated with the matter frame and simply replace $a \to i$ where $i = 1 - 3$. We then obtain the equations that govern transverse perturbations (for which the displacement is orthogonal to the direction of propagation) by taking the curl ($\epsilon^{jki}\nabla_k$) of the equations in the usual way. Hence, we arrive at

$$m_\star\dot{U}^i - \frac{1}{\tau}\dot{Q}^i = 0 \tag{42}$$

and

$$m_\star\dot{Q}^i + (p + \rho)Q^i = 0, \tag{43}$$

where we have defined

$$U^i = \epsilon^{ijk}\nabla_j\delta u_k \qquad \text{and} \qquad Q^i = \epsilon^{ijk}\nabla_j\delta q_k, \tag{44}$$

and

$$m_\star = n\left(\mu - \frac{\alpha\kappa T}{\tau}\right) = p + \rho - \frac{\kappa T}{\tau}. \tag{45}$$

Assuming that the perturbations depend on time as $e^{i\omega t}$, where t is the time coordinate associated with the matter frame, we arrive at the dispersion relation

$$i\omega\left[(p+\rho)(1+i\omega\tau) - i\omega\kappa T\right] = 0. \tag{46}$$

Obviously $\omega = 0$ is a solution. The second root is

$$\omega = \frac{i(p+\rho)}{m_\star\tau}. \tag{47}$$

This result shows that the thermal relaxation time τ is essential in order for the system to be stable. We need $m_\star > 0$, i.e. the relaxation time must be such that

$$\tau > \frac{\kappa T}{p+\rho}. \tag{48}$$

The analysis shows that Eckart's model (for which $\tau = 0$) is inherently unstable. Moreover, the constraint on the relaxation time agrees with one of the conditions obtained by Olson and Hiscock [25] (cf, their equation (41)), representing the inviscid limit of the exhaustive analysis of the Israel–Stewart model of Hiscock and Lindblom [19]. We may also note that the condition given in equation (43) of [25] simply leads to the weaker requirement $\tau \geqslant 0$.

The physical origin of the transverse instability can be understood if we rewrite (40) and (41) as

$$m_\star\delta\dot{u}_a + \perp_a^b \nabla_b\delta p - \frac{1}{\tau}\left(\delta q_a + \kappa \perp_a^b \nabla_b\delta T\right) = 0, \tag{49}$$

and

$$m_\star\delta\dot{q}_a + (p+\rho)\left(\delta q_a + \kappa \perp_a^b \delta T\right) - \kappa T \perp_a^b \nabla_b\delta p = 0. \tag{50}$$

These relations show that $m_\star$ plays the role of an 'effective' inertial mass (density). The importance of this quantity has been discussed in a series of papers by Herrera and collaborators [29–31], especially in the context of gravitational collapse. Basically, the instability of the Eckart formulation is due to the inertial mass of the fluid becoming negative. Once this happens the pressure gradient no longer provides a restoring force, rather it would tend to push the system further away from equilibrium. This is a run-away process associated with exponential growth of the perturbations. Ultimately, the instability is due to the inertia of heat; an unavoidable consequence of the equivalence principle (heat carries energy, which means that it can be associated with an effective mass [32]). Condition (48) may seem rather extreme (Hiscock and Lindblom [20] quote an instability timescale of 10^{-35} s for water at 300 K), but it sets a sharp lower limit for the thermal relaxation in physical systems. A system with faster thermal relaxation cannot settle down to equilibrium. However, it may still be reasonable to ask if a system may evolve in such a way that it enters the unstable regime (in the way discussed in [29, 30]). Given our assumed equilibrium the present formulation does not allow us to consider this question, but it seems clear that if a system were to evolve in that way, then one would need a fully nonlinear analysis to determine the consequences.

3.3. The longitudinal problem

The transverse problem is relatively simple since there are no corresponding restoring forces in a simple fluid problem (these require rotation, elasticity, the presence of a magnetic field, etc). When we turn to the longitudinal case the situation changes. In a perfect fluid longitudinal perturbations propagate as sound waves, and when we add complexity to the model the dispersion relation can get very complicated.

To analyse the longitudinal case, we take the divergence of (40) and (41). This leads to

$$(p + \rho)\partial_t \nabla_i \delta u^i + \nabla^2 \delta p + \partial_t \nabla_i \delta q^i = 0, \tag{51}$$

and

$$\tau \partial_t \nabla_i \delta q^i + \nabla_i \delta q^i + \kappa \nabla^2 \delta T + \kappa T \partial_t \nabla_i \delta u^i = 0. \tag{52}$$

We also need the conservation laws which take the form

$$\nabla_i \delta u^i = -\frac{1}{n}\partial_t \delta n, \tag{53}$$

$$\nabla_i \delta q^i = -nT\partial_t \delta\bar{s}, \tag{54}$$

where we have defined the specific entropy; $\bar{s} = s/n$. To make progress we need to, first of all, decide what variables to work with, and secondly we need an equation of state for matter. In the following, we will opt to work with the perturbed densities δn and $\delta\bar{s}$. Keeping in mind that we are only retaining first-order quantities, we then have

$$\delta p = \left(\frac{\partial p}{\partial n}\right)_{\bar{s}} \delta n + \left(\frac{\partial p}{\partial \bar{s}}\right)_n \delta\bar{s}, \tag{55}$$

and

$$\delta T = \left(\frac{\partial T}{\partial n}\right)_{\bar{s}} \delta n + \left(\frac{\partial T}{\partial \bar{s}}\right)_n \delta\bar{s}. \tag{56}$$

It is also useful to recall that the temperature represents the 'entropy chemical potential', which is defined by

$$T = \left(\frac{\partial \rho}{\partial s}\right)_n, \tag{57}$$

where $\rho = \rho(n, s)$ represents the equation of state. This immediately implies that

$$\left(\frac{\partial T}{\partial n}\right)_s = \left(\frac{\partial \mu}{\partial s}\right)_n, \tag{58}$$

i.e. we can reduce the number of thermodynamic quantities. In order to make contact with (potential) observations, it is natural to work with (i) the adiabatic speed of sound

$$c_s^2 = \left(\frac{\partial p}{\partial \rho}\right)_{\bar{s}} = \frac{n}{p + \rho}\left(\frac{\partial p}{\partial n}\right)_{\bar{s}}, \tag{59}$$

(ii) the heat capacity at fixed volume

$$c_v = T\left(\frac{\partial \bar{s}}{\partial T}\right)_n \tag{60}$$

and (iii)

$$\alpha_s = \frac{n}{T}\left(\frac{\partial T}{\partial n}\right)_{\bar{s}} = \frac{T}{n}\left(\frac{\partial p}{\partial \bar{s}}\right)_n. \tag{61}$$

For future reference, it is also useful to note the identity (cf, equation (96) in Hiscock and Lindblom [19])

$$\frac{1}{c_v} - \frac{1}{c_p} = \frac{n^3}{T(p + \rho)c_s^2}\left(\frac{\partial T}{\partial n}\right)_{\bar{s}}^2 = \frac{nT}{(p + \rho)c_s^2}\alpha_s, \tag{62}$$

where c_p is the heat capacity at fixed pressure.

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If we (again) focus on plane-wave solutions such that the perturbations behave as $\exp(\mathrm{i}\omega t + \mathrm{i}k_j x^j)$, and introduce the phase velocity $\sigma = \omega/k$, then the above relations lead to the coupled equations

$$\frac{p + \rho}{n}\left(\sigma^2 - c_s^2\right)\delta n + nT(\sigma^2 - \alpha_s)\delta\bar{s} = 0 \tag{63}$$

and

$$\frac{\kappa T}{n}\left(\sigma^2 - \alpha_s\right)\delta n + T\left(n\tau\sigma^2 - \frac{\mathrm{i}\sigma n}{k} - \frac{\kappa}{c_v}\right)\delta\bar{s} = 0. \tag{64}$$

From these results it is easy to see that the generic dispersion relation will be a quartic in σ. This is exactly what one would expect for a physical system. As we have already mentioned, the formulation must accommodate the presence of 'second sound' which has been experimentally verified for both superfluids and low-temperature solids. Working out the explicit dispersion relation, we find

$$m_\star \tau\sigma^4 - \frac{\mathrm{i}(p + \rho)}{k}\sigma\left(\sigma^2 - c_s^2\right) - \left[(p + \rho)\left(\frac{\kappa}{nc_v} + c_s^2\tau\right) - 2\kappa T\alpha_s\right]\sigma^2$$
$$+ \kappa\left[\frac{p + \rho}{n}\frac{c_s^2}{c_v} - T\alpha_s^2\right] = 0. \tag{65}$$

This expression is still too complicated for us to be able to make definite statements about the solutions, without making further assumptions. The most direct strategy would be to consider an explicit equation of state, work out the relevant thermodynamics quantities and then solve (65) (probably numerically). This would allow us to establish whether the considered model is stable and causal. This route is, however, not particularly attractive given the need to introduce an explicit model. If we want to continue to consider the problem at some level of generality, then we need to resort to approximations. As we will see, this is a very instructive route.

In order to simplify the analysis, we will consider the long- and short-wavelength limits of the problem. The results we obtain in these limits provide useful illustrations of the key features. At the same time, we should keep in mind that both cases are somewhat 'artificial'. First of all, fluid dynamics is, fundamentally, an effective long-wavelength theory in the sense that it arises from an averaging over a large number of individual particles (constituting each fluid element). In effect, the model only applies to phenomena on scales much larger than (say) the interparticle distance. However, the infinite-wavelength limit represents a uniform system, which is artificial since real physical systems tend to be finite. Moreover, as we will not account explicitly for gravity we can only consider scales on which spacetime can be considered flat. While the plane-wave analysis holds on arbitrary scales in special relativity, a curved spacetime introduces a cut-off lengthscale beyond which the analysis is not valid. The theoretical framework is valid in general, but on larger scales we would have to consider also the perturbed Einstein equations.

3.4. The long-wavelength problem

Let us first consider the long-wavelength, $k \to 0$, problem. This represents the true hydrodynamic limit, and it is easy to see that there are two sound-wave solutions and two modes that are predominantly diffusive. The sound-wave solutions take the form

$$\sigma \approx \pm c_s\left[1 \pm \mathrm{i}\frac{\kappa T}{2(p + \rho)c_s^3}\left(c_s^2 - \alpha_s\right)^2 k\right]. \tag{66}$$

These solutions are clearly stable, since $\mathrm{Im}\,\sigma > 0$. Using the Maxwell relations listed by Hiscock and Lindblom [19], we can show that this result agrees with equation (40) from [20]. Moreover, our result simplifies to (using (62))

$$\mathrm{Im}\,\sigma \approx \frac{\kappa}{2n}\left(\frac{1}{c_v} - \frac{1}{c_p}\right), \tag{67}$$

in the limit where $|\alpha_s| \gg c_s^2$, which is relevant since $c_s^2 \sim p/\rho$ becomes small in the non-relativistic limit. Indeed, we find that (67) agrees with the standard result for sound absorption in a heat-conducting medium [33].

In addition to the sound waves, we have a slowly damped solution

$$\sigma \approx \mathrm{i}\kappa\left[\frac{1}{nc_v} - \frac{T\alpha_s^2}{(p+\rho)c_s^2}\right] = \frac{\mathrm{i}\kappa}{nc_p}. \tag{68}$$

This is the classic result for thermal diffusion.

Finally, the system has a fast decaying solution,

$$\sigma \approx \frac{\mathrm{i}(p+\rho)}{m_\star k \tau}. \tag{69}$$

Under most circumstances, this root is likely to decay too fast to be observable (recall (48)). When this is the case, the model reproduces the standard 'Rayleigh–Brillouin spectrum' with two sound peaks symmetrically placed with respect to the broad diffusion peak at zero frequency [33, 34].

3.5. Short-wavelength stability and causality

Different aspects of the problem are probed in the short-wavelength limit. Letting $k \to \infty$ we see[4] that (65) reduces to a quadratic for σ^2. This allows us to write down the solutions in closed form. Hence, it is relatively straightforward to establish the conditions required for the stability of the system in this limit. For infinitesimal wavelengths, we have

$$A\sigma^4 - B\sigma^2 + C = 0, \longrightarrow \sigma_\pm^2 = \frac{1}{2A}[B \pm (B^2 - 4AC)^{1/2}], \tag{70}$$

where

$$A = m_\star\tau > 0, \tag{71}$$

in order for transverse perturbations to be stable. We also have

$$B = (p+\rho)\left(\frac{\kappa}{nc_v} + c_s^2\tau\right) - 2\kappa T\alpha_s \tag{72}$$

and

$$C = \kappa\left[\frac{p+\rho}{n}\frac{c_s^2}{c_v} - T\alpha_s^2\right] = \kappa\left(\frac{p+\rho}{n}\right)\frac{c_s^2}{c_p} > 0. \tag{73}$$

In order to guarantee stability for longitudinal perturbations, we need σ^2 to be real and positive. Given the quadratic formula and the fact that $A > 0$, this implies that we must have $B^2 - 4AC > 0$. After some algebra, this leads to

$$\left(c_s^2\tau - \frac{\kappa}{nc_v} - \frac{2\kappa T\alpha_s}{p+\rho}\right)^2 + \frac{4\kappa T\alpha_s^2}{p+\rho}\left(\tau - \frac{\kappa T}{p+\rho}\right) + \frac{4\kappa^2 T}{(p+\rho)nc_v}\left(c_s^2 - 2\alpha_s\right) > 0. \tag{74}$$

---

[4] In order to be precise, we should state more clearly in what sense k is large. However, the validity of the model, i.e. the physical scale(s) that the wavelength should be compared to are quite easy to work out from the final results, should one want to do so. Hence, we will not state the detailed conditions here.

The first two terms are positive, as long as (48) is satisfied. Hence, the condition is guaranteed to be satisfied as long as $c_s^2 > \alpha_s$. As discussed later, this is certainly the case for degenerate matter. In cases where this simple condition is not satisfied, (74) provides a complicated constraint on the relaxation time.

Finally, we must also have $B > 0$, which can be expressed as

$$\tau > \frac{\kappa}{c_s^2} \left[\frac{2T}{p+\rho} \alpha_s - \frac{1}{nc_v} \right]. \tag{75}$$

This condition is identical to that given in equation (146) of [19] (obtained in the limit where $\alpha_i \to 0$ and $1/\beta_0$ and $1/\beta_2$ both also vanish, cf [2, 35]).

Let us now consider finite wavelengths. Letting $\sigma = \sigma_\pm + \sigma_1/k$, where $\sigma_\pm$ solve (70), and linearizing in $1/k$, we find that

$$\sigma_1 = \frac{\mathrm{i}(p+\rho)}{2} \left(\frac{\sigma_\pm^2 - c_s^2}{2A\sigma_\pm^2 - B} \right). \tag{76}$$

Since all quantities in this expression are already constrained to be real, we need $\mathrm{Im}\,\sigma_1 \geqslant 0$ (for real k) in order for the system to be stable. From (70) we see that

$$2A\sigma_\pm^2 - B = \pm \left| B^2 - 4AC \right|^{1/2}. \tag{77}$$

This leads to the final condition

$$\sigma_-^2 \leqslant c_s^2 \leqslant \sigma_+^2. \tag{78}$$

It is worth noting that this result is consistent with the notion that 'mode-mergers' signal the onset of instability [21].

As the waves in the system must remain causal, we should insist that $\sigma^2 < 1$. To ensure that this is the case, we adapt the strategy used by Hiscock and Lindblom [19]. As (70) is a quadratic for σ^2 we can ensure that the roots are confined to the interval $0 < \sigma^2 < 1$ (noting first of all that the roots are real since (74) is satisfied). Given that B and C are both positive, the roots must be such that $\sigma^2 > 0$. Meanwhile, we can constrain the roots to $\sigma^2 < 1$ by insisting that

$$A - B + C > 0 \tag{79}$$

and

$$A - 2B > 0. \tag{80}$$

Combining these inequalities with the positive discriminant, we can show that $A > B/2 > C$. The first of the two conditions can be written as

$$\left(1 - c_s^2\right) \left[\tau - \frac{\kappa}{nc_v} \right] > \frac{\kappa T (1 - \alpha_s)^2}{p + \rho} > 0. \tag{81}$$

Now, when combined with causality the condition (78) requires that $c_s^2 \leqslant \sigma_+^2 < 1$. In other words, we must have $c_s^2 < 1$, which means that (81) implies that

$$\tau > \frac{\kappa}{nc_v}. \tag{82}$$

Comparing to the results of Hiscock and Lindblom [19], we recognize (81) as their $\Omega_3 > 0$ condition (it is also equation (4) of Herrera and Martinez [30]), while (82) corresponds to $\Omega_6 > 0$.

Meanwhile, condition (80) can be written as

$$\left(2 - c_s^2\right)\tau > \frac{\kappa}{nc_v} + \frac{2\kappa T}{p + \rho}(1 - \alpha_s), \tag{83}$$

corresponding to equation (148) of Hiscock and Lindblom. Finally, $A > C$ leads to

$$\tau > \frac{\kappa T}{p + \rho} + \frac{\kappa c_s^2}{n c_p}.$$

(84)

This corresponds to equation (3) of Herrera and Martinez [30], which derives from equation (147) of Hiscock and Lindblom [19].

This completes the analysis of the stability and causality of the system. We have arrived at a set of conditions on the thermal relaxation time (and related them to results in the existing literature). As long as these conditions are satisfied, the solutions to the problem should be well behaved.

3.6. The emergence of second sound

So far we have considered the conditions that must be satisfied by a thermodynamical model in order to ensure the stability and causality of both transverse and longitudinal waves. To complete the analysis of the problem, we will now consider the nature of the solutions. Since the phase velocity σ is obtained from a quartic, we know that the problem has two (wave) degrees of freedom. This accords with the experience from superfluid systems and experimental evidence for heat propagating as waves in low-temperature solids. One of the solutions should be associated with the usual 'acoustic' sound while the second degree of freedom will lead to a 'second sound' for heat. We want to demonstrate how these features emerge within our model.

In order to explore these issues, it is natural to consider the long relaxation time limit. Taking τ to be large, the solutions to (70) take the form (up to, and including, order $1/\tau$ terms)

$$\sigma_+^2 \approx c_s^2 \left[1 + \frac{\kappa T}{(p + \rho)\tau} \left(1 + \frac{\alpha_s^2}{c_s^4} \right) \right],$$

(85)

which could be rewritten using (62), and

$$\sigma_-^2 \approx \frac{\kappa}{n \tau c_p}.$$

(86)

The first of these solutions clearly represents the usual sound, while the other solution provides the second sound. In the latter case, the deduced speed is exactly what one would expect [16]. It is easy to see that the first root will satisfy (78), and the associated roots will be stable in the long relaxation time limit. The second solution leads to stable roots provided

$$\tau \geqslant \frac{\kappa}{n c_p c_s^2}.$$

(87)

Basically, the finite-wavelength condition implies that the second sound must propagate slower than the first sound. This is, indeed, what is measured in physical systems (like superfluid helium). Moreover, it is easy to see that this condition must be satisfied in order for the long relaxation time approximation to be valid.

As a useful illustration of the properties of the model, let us consider the particular case of degenerate matter. In this case, which relates to electrons in both metals and white dwarfs (and also neutrons and protons in neutron stars), the two specific heats are almost identical,

$$\frac{c_v}{c_p} \approx 1 + O \left(\frac{k_B T}{\epsilon_F} \right)^2,$$

(88)

where k_B is Boltzmann's constant and ϵ_F is the relevant Fermi energy [36]. This means that, for temperatures significantly below the Fermi temperature, we can accurately assume that

$\alpha_s \approx 0$. If we also assume that $\tau \gg \kappa T/(p + \rho)$, then the dispersion relation factorizes and we have

$$\left(\sigma^2 - c_s^2\right)\left(\tau\sigma^2 - \frac{\mathrm{i}\sigma}{k} - \frac{\kappa}{nc_v}\right) = 0. \tag{89}$$

That is, the four roots are

$$\sigma = \pm c_s \tag{90}$$

and

$$\sigma = \frac{\mathrm{i}}{2k\tau}\left[1 \pm \left(1 - \frac{4\kappa\tau}{nc_v}k\right)^{1/2}\right]. \tag{91}$$

The character of these roots is illustrated in figure 1. We see that the ordinary sound exists at all wavelengths. Meanwhile, at long wavelengths (small k) the remaining two roots are exponentially damped, i.e. diffusive in character. One root has a relatively slow decay, corresponding to the expected thermal diffusion, while the other root decays rapidly. Below a critical lengthscale (corresponding to $k = 10$ in figure 1) the second sound emerges as a result of the finite thermal relaxation time τ. For very short lengthscales, heat signals will propagate as waves. However, as is clear from (91) these solutions are always damped. In order to 'propagate', the real part of the wave frequency must exceed the imaginary part (so that several cycles are executed before the motion is damped out). This boils down to the second sound propagating only for

$$k \gg \left(\frac{\kappa\tau}{nc_v}\right)^{-1/2}. \tag{92}$$

This result is interesting if we consider systems that become superfluid (see [37] for an interesting discussion of models for relativistic superfluids). Suppose we consider a system which starts out in the diffusive regime (e.g. helium above the superfluid transition temperature). When the system is cooled down through the relevant transition temperature, the (non-momentum conserving) particle collisions that give rise to κ are suppressed. In effect, the critical value of k decreases and the system may enter the regime where the second sound can propagate at macroscopic scales. The change to the basic nature of heat propagation is easily understood.

At the end of the day, this relatively simple model demonstrates how the behaviour of a given physical system depends on the balance between the characteristic timescales. Obviously, we also need to keep in mind that real systems impose restrictions both for large k, as the fluid model breaks down when we approach the interparticle distance scale, and small k, when the wavelength becomes larger than the size of the system.

4. Final remarks

We set out to derive a consistent first-order model for relativistic heat conduction, in such a way that the theory remains cognizant of its higher order origins. As we have demonstrated, this leads to a model that retains the thermal relaxation that is necessary if we want the problem to remain causal. What have we learned from this exercise? First of all, we have illustrated the problems associated with the original first-order models, e.g. that of Eckart [17]. The conclusions are, obviously, not new but the discussion should lay to rest any suggestions that the coupling of the four-acceleration to the heat flux is (somehow) problematic [26–28]. By considering the waves present in the new model, we have established that the system is both stable and causal provided that some seemingly natural conditions are satisfied. This
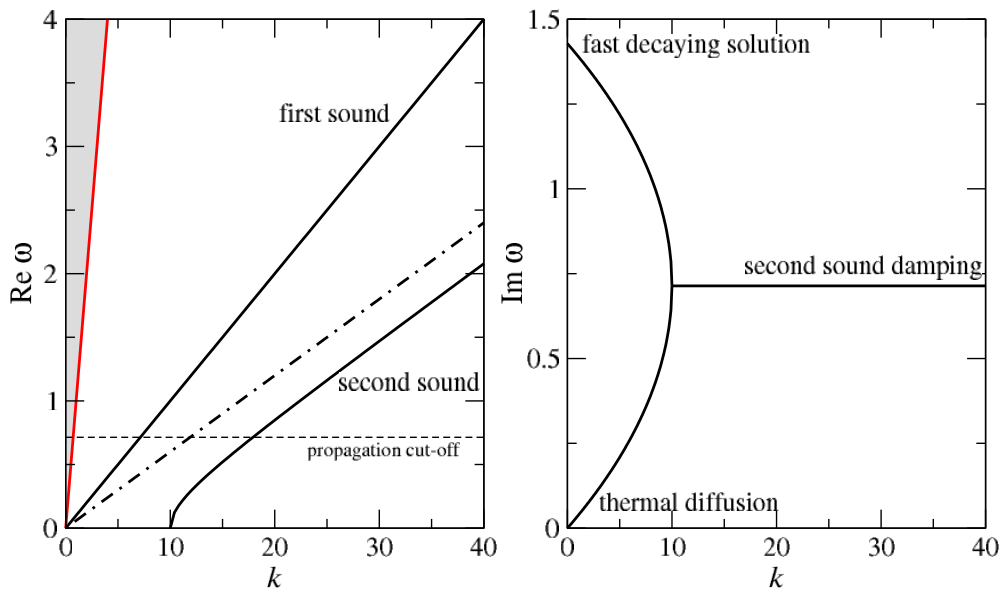
Figure 1. This figure provides an illustration of the qualitative nature of the behaviour of heat conducting degenerate matter (as discussed in the main text), based on the consistent first-order relativistic model. The parameters have been chosen in such a way that the speed of sound is 10% of the speed of light, while the second sound (at short wavelengths, large k) propagates at $1/\sqrt{3}$ of this. The phase velocity of the waves is $\sigma = \mathrm{Re}\,\omega/k$ (cf left panel). The thermal relaxation time τ has been chosen such that the critical wavenumber at which the second sound emerges is $k = 10$. At lengthscales larger than this, the corresponding roots are diffusive (have purely imaginary frequency), and in the very long wavelength limit ($k \to 0$) we retain the expected thermal diffusion. The damping time follows from $1/\mathrm{Im}\,\omega$ (cf right panel). We also indicate the acausal region (grey area). The illustrated example is clearly both stable and causal.

(This figure is in colour only in the electronic version)

conclusion accords with the classic work by Hiscock and Lindblom [19, 20] (see also Olson and Hiscock [25]). In fact, the conditions we have arrived at reproduce their key results. However, our analysis adds to previous work by discussing the emergence of the second sound and the nature of the associated solutions. This is a key point, especially if we are interested in relativistic superfluid systems. The analysis also demonstrates the intricate nature of these problems. It is easy to see how a model that may fail one, or several, of the derived conditions in some regime may nevertheless be valid for a different range of parameters. Hence, one really should consider the applicability of the chosen theory on a case-by-case basis. This is probably no more than should be expected from a phenomenological model.

Our results represent useful progress in this problem area, but one could obviously develop the theory further. A natural step would be to consider the various constraints that we have derived for detailed equations of state, e.g., matter coupled to phonons. It would also be interesting to consider applications of the first-order construction. While the model is restricted in the sense that it does not account for non-adiabatic effects, there is an exciting range of possible applications in astrophysics, cosmology and high-energy physics. Particularly interesting questions concern to what extent second sound effects are relevant in relativistic systems and the difference between first-order results and the, considerably more complex, second-order theories.

Acknowledgments

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